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Interpolating Rational Splines in Three Dimensional Space*

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Abstract Let a polyhedron in three dimensional space be decomposed into tetrahedral cells by a certain partition. In this paper efforts are made on assigning appropriate nodes along the edges of every tetrahedron and characterizing interpolation data that determine a unique rational function of type (1, 1), which is nonsingular in the corresponding tetrahedron. By constructing suitable basis functions and restricting the interpolation data, necessary and sufficient conditions for the existence of rational splines with C^0 as well as C^1 smoothness are formulated respectively.

Keywords partition, interpolation, rational spline.

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1. Introduction

Bivariate splines, which are composed of piecewise polynomials over collections of convex polygons, usually triangles or rectangles, have been thoroughly studied by various authors (see, e.g., [3], [4] and the bibliography therein). It is somewhat difficult, however, to apply the techniques developed in polynomial spline theory to the investigation of multivariate rational splines since there are many undetermined factors in treating nonlinear problems. As we know that the first derivative of a rational function of (n,m)- type is of $(n+m-1, 2m)$- type, which leads to difficulties in patching together piecewise rational functions as smoothly as possible, moreover in most cases a rational function is required to be nonsingular in the corresponding domain, which results in complications of treating a rational function with higher- degree denominator. By means of Wachspress' rational wedge functions ([1]) the author ([5]) found an effective approach to the construction of bivariate nonsingular rational splines with low orders under triangular and quadrilateral partitions respectively. This paper aims at finding out the conditions for the existence of interpolating nonsingular rational splines in three dimensional space.

Suppose that V is a polyhedron in three dimensional space R^3, which is divided into tetrahedral cells by a partition Ω containing N interior edges $E_1, E_2, \cdots, E_N$. Denote by $B(E_i)$ the union of all the cells such that E_i acts as a common edge of them. Assume that $B(E_i)$

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contains M_i tetrahedral cells which are denoted as $\Omega_1^i, \Omega_2^i, \cdots, \Omega_{M_i}^i$ in counterclockwise order. For each cell Ω_k^i we label its vertices and some point on each edge as $p^{k,i}$, $p = 1, 2, \cdots, 10$ such that the labels on the triangular planes opposite to vertices $1^{k,i}$, $2^{k,i}$, $3^{k,i}$ and $4^{k,i}$ are respectively $2^{k,i}$, $9^{k,i}$, $3^{k,i}$, $7^{k,i}$, $4^{k,i}$ and $6^{k,i}$; $1^{k,i}$, $10^{k,i}$, $3^{k,i}$, $7^{k,i}$, $4^{k,i}$ and $5^{k,i}$; $1^{k,i}$, $8^{k,i}$, $2^{k,i}$, $6^{k,i}$, $4^{k,i}$ and $5^{k,i}$; and $1^{k,i}$, $8^{k,i}$, $2^{k,i}$, $9^{k,i}$, $3^{k,i}$ and $10^{k,i}$. Moreover we make the convention that vertices $2^{k,i}$ and $3^{k+1,i}$ correspond to the same triangular plane, i.e., $1^{k,i} = 1^{k+1,i}$, $3^{k,i} = 2^{k+1,i}$, $4^{k,i} = 4^{k+1,i}$, $5^{k,i} = 5^{k+1,i}$, $7^{k,i} = 6^{k+1,i}$ and $10^{k,i} = 8^{k+1,i}$ (as shown in Fig. 1).

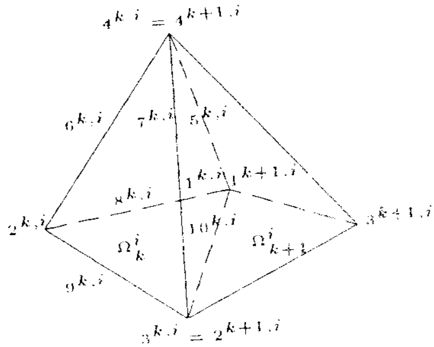

Fig. 1

We say a rational function in R^3 is of (m, n)- type if its numerator and denominator are polynomials in x, y and z of degree m and n respectively, and denote by $R_{m,n}(x, y, z)$ the set of all the rational functions of (m, n)- type.

Definition 1 *If the rational function $R^{k,i}(x, y, z)$ in $R_{1,1}(x, y, z)$, which has no singular points in Ω_k^i, satisfies*

$$R^{k,i}(p^{k,i}) = \alpha_p^{k,i} = u_p^{k,i}/v_p^{k,i}, \quad p = 1, 2, \cdots, 10; \quad k = 1, 2, \cdots, M_i; \quad i = 1, 2, \cdots, N, \tag{1.1}$$

$$R^{k,i}(x, y, z) - R^{k+1,i}(x, y, z) = O((l_k(x, y, z))^t), \quad t = 1, 2, \tag{1.2}$$

where $u_p^{k,i}/v_p^{k,i}$ is a fraction given at $p^{k,i} = (x_p^{k,i}, y_p^{k,i}, z_p^{k,i})$ with the convention that $v_p^{k,i} > 0$ and $v_p^{k,i} = 1$ if $\alpha_p^{k,i} = 0$, and $l_k(x, y, z)$ denotes the linear form whose locus contains $\Omega_k^i \cap \Omega_{k+1}^i$, then we call the union of the M_i rational functions an incident interpolating rational spline associated with E_i of $(1, 1)$- type with C^{t-1}-smoothness and denote it by $SR_{1,1}^{t-1}(x, y, z; B(E_i), \Omega)$.

Definition 2 *If $B(E_i) \cap B(E_j) \supset \Omega_m^i = \Omega_n^j$ implies $R^{m,i}(x, y, z) = R^{n,j}(x, y, z)$, then we call $\bigcup_{i=1}^N SR_{1,1}^{t-1}(x, y, z; B(E_i), \Omega)$ the interpolating rational spline defined in D of $(1, 1)$-type with C^{t-1}-smoothness and denote it by $SR_{1,1}^{t-1}(x, y, z; V, \Omega)$ or $SR_{1,1}^{t-1}(V, \Omega)$ for short.*

For simplicity the sub-or superscripts k, i in what follows will be all or partially dropped unless they are necessary and so will be done with later introduced notations whereever the context makes them clear.

2 C^0 rational splines

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Denote by $d(m, n)$ the distance between two points m and n and let

$$F_{m,n} = \frac{\alpha_m - \alpha_n}{d(m, n)}. \tag{2.1}$$

Theorem 1 *The necessary and sufficient conditions for the existence of* $SR^0_{1,1}(\Omega, V)$ *satisfying* (1.1) *are the following*

$$F_{1,8}F_{2,9}F_{3,10} = F_{8,2}F_{9,3}F_{10,1}, \tag{2.2}$$

$$F_{1,8}F_{2,6}F_{4,5} = F_{8,2}F_{6,4}F_{5,1}, \tag{2.3}$$

$$F_{3,10}F_{1,5}F_{4,7} = F_{10,1}F_{5,4}F_{7,3}, \tag{2.4}$$

$$F_{2,6}F_{4,7}F_{3,9} = F_{6,4}F_{7,3}F_{9,2}, \tag{2.5}$$

$$\min(\alpha_1, \alpha_4) < \alpha_5 < \max(\alpha_1, \alpha_4), \tag{2.6}$$

$$\min(\alpha_2, \alpha_4) < \alpha_6 < \max(\alpha_2, \alpha_4), \tag{2.7}$$

$$\min(\alpha_3, \alpha_4) < \alpha_7 < \max(\alpha_3, \alpha_4), \tag{2.8}$$

$$\min(\alpha_1, \alpha_2) < \alpha_8 < \max(\alpha_1, \alpha_2), \tag{2.9}$$

$$\min(\alpha_2, \alpha_3) < \alpha_9 < \max(\alpha_2, \alpha_3), \tag{2.10}$$

$$\min(\alpha_1, \alpha_3) < \alpha_{10} < \max(\alpha_1, \alpha_3), \tag{2.11}$$

$$\alpha_1^k = \alpha_1^{k+1}, \quad \alpha_3^k = \alpha_2^{k+1}, \alpha_4^k = \alpha_4^{k+1}, \tag{2.12}$$

$$\alpha_5^k = \alpha_5^{k+1}, \quad \alpha_7^k = \alpha_6^{k+1}, \quad \alpha_{10}^k = \alpha_8^{k+1}. \tag{2.13}$$

Proof Let us introduce the following notations

$$D(i, j, k, \bullet) = \begin{vmatrix} 1 & 1 & 1 & 1 \\ x_i & x_j & x_k & x \\ y_i & y_j & y_k & y \\ z_i & z_j & z_k & z \end{vmatrix}, \tag{2.14}$$

$$\begin{aligned} W_1(\bullet) &= \frac{D(2, 3, 4, \bullet)}{D(2, 3, 4, 1)}, \quad W_2(\bullet) = \frac{D(3, 4, 1, \bullet)}{D(3, 4, 1, 2)}, \\ W_3(\bullet) &= \frac{D(4, 1, 2, \bullet)}{D(4, 1, 2, 3)}, \quad W_4(\bullet) = \frac{D(1, 2, 3, \bullet)}{D(1, 2, 3, 4)}. \end{aligned} \tag{2.15}$$

Obviously we have

$$W_p(q) = \delta_{pq}, \quad p, q = 1, 2, 3, 4, \tag{2.16}$$

where δ_{pq} is the Kronecker symbol. Let

$$R(x, y, z) = \frac{\sum_{p=1}^{4} C_p u_p W_p(x, y, z)}{\sum_{p=1}^{4} C_p v_p W_p(x, y, z)}, \tag{2.17}$$

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then it follows from (2.15) that $R(p) = \alpha_p$ holds true for $p = 1, 2, 3, 4$. Now we construct the coefficients C_p's such that

$$R(p) = \alpha_p, \quad p = 1, 2, \cdots 10. \tag{2.18}$$

By (2.14) and (2.15) we have

$$\begin{aligned} W_1(6) &= W_1(7) = W_1(9) = 0, \\ W_2(5) &= W_2(7) = W_2(10) = 0, \\ W_3(5) &= W_3(6) = W_3(8) = 0, \\ W_4(8) &= W_4(9) = W_4(10) = 0, \end{aligned} \tag{2.19}$$

because W_i vanishes on the line connecting points distinct from point i, therefore (2.18) leads to the conditions

$$\begin{aligned} &(u_1v_5 - u_5v_1)W_1(5)C_1 + (u_4v_5 - u_5v_4)W_4(5)C_4 = 0, \\ &(u_2v_6 - u_6v_2)W_2(6)C_2 + (u_4v_6 - u_6v_4)W_4(6)C_4 = 0, \\ &(u_3v_7 - u_7v_3)W_3(7)C_3 + (u_4v_7 - u_7v_4)W_4(7)C_4 = 0, \\ &(u_1v_8 - u_8v_1)W_1(8)C_1 + (u_2v_8 - u_8v_2)W_2(8)C_2 = 0, \\ &(u_2v_9 - u_9v_2)W_2(9)C_2 + (u_3v_9 - u_9v_3)W_3(9)C_3 = 0, \\ &(u_1v_{10} - u_{10}v_1)W_1(10)C_1 + (u_3v_{10} - u_{10}v_3)W_3(10)C_3 = 0. \end{aligned} \tag{2.20}$$

Making use of the following basic facts

$$D(i, j, k, \alpha s + \beta t) = \alpha D(i, j, k, s) + \beta D(i, j, k, t) \tag{2.21}$$

where $\alpha + \beta = 1$ and

$$\begin{aligned} 5 &= \frac{d(1,5)}{d(1,4)}4 + \frac{d(4,5)}{d(1,4)}1, & 6 &= \frac{d(2,6)}{d(2,4)}4 + \frac{d(4,6)}{d(2,4)}2, \\ 7 &= \frac{d(3,7)}{d(3,4)}4 + \frac{d(4,7)}{d(3,4)}3, & 8 &= \frac{d(2,8)}{d(1,2)}1 + \frac{d(1,8)}{d(1,2)}2, \\ 9 &= \frac{d(3,9)}{d(2,3)}2 + \frac{d(2,9)}{d(2,3)}3, & 10 &= \frac{d(3,10)}{d(1,3)}1 + \frac{d(1,10)}{d(1,3)}3, \end{aligned} \tag{2.22}$$

we obtain

$$\begin{aligned} W_1(5) &= \frac{D(2,3,4,5)}{D(2,3,4,1)} = \frac{d(4,5)}{d(1,4)}, & W_4(5) &= \frac{D(1,2,3,5)}{D(1,2,3,4)} = \frac{d(1,5)}{d(1,4)}, \\ W_2(6) &= \frac{D(3,4,1,6)}{D(3,4,1,2)} = \frac{d(4,6)}{d(2,4)}, & W_4(6) &= \frac{D(1,2,3,6)}{D(1,2,3,4)} = \frac{d(2,6)}{d(2,4)}, \\ W_3(7) &= \frac{D(4,1,2,7)}{D(4,1,2,3)} = \frac{d(4,7)}{d(3,4)}, & W_4(7) &= \frac{D(1,2,3,7)}{D(1,2,3,4)} = \frac{d(3,7)}{d(3,4)}, \\ W_1(8) &= \frac{D(2,3,4,8)}{D(2,3,4,1)} = \frac{d(2,8)}{d(1,2)}, & W_2(8) &= \frac{D(3,4,1,8)}{D(3,4,1,2)} = \frac{d(1,8)}{d(1,2)}, \\ W_2(9) &= \frac{D(3,4,1,9)}{D(3,4,1,2)} = \frac{d(3,9)}{d(2,3)}, & W_3(9) &= \frac{D(4,1,2,9)}{D(4,1,2,3)} = \frac{d(2,9)}{d(2,3)}, \\ W_1(10) &= \frac{D(2,3,4,10)}{D(2,3,4,1)} = \frac{d(3,10)}{d(1,3)}, & W_3(10) &= \frac{D(4,1,2,10)}{D(4,1,2,3)} = \frac{d(1,10)}{d(1,3)}. \end{aligned} \tag{2.23}$$

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Let

$$G_{m,n} = \frac{u_m v_n - u_n v_m}{d(m,n)} = v_m v_n F_{m,n}, \tag{2.24}$$

then (2.20) is equivalent to

$$\begin{aligned} &G_{1,5}C_1 + G_{4,5}C_4 = 0, \\ &G_{2,6}C_2 + G_{4,6}C_4 = 0, \\ &G_{3,7}C_3 + G_{4,7}C_4 = 0, \\ &G_{1,8}C_1 + G_{2,8}C_2 = 0, \\ &G_{2,9}C_2 + G_{3,9}C_3 = 0, \\ &G_{1,10}C_1 + G_{3,10}C_3 = 0. \end{aligned} \tag{2.25}$$

Clearly a nontrivial $R(x,y,z)$ satisfying (2.18) is uniquely determined if and only if every one of fifteen 4×4 submatrices contained in the 6×4 matrix with respect to coefficients C_1, C_2, C_3 and C_4 has rank three, which yields

$$G_{1,8}G_{2,9}G_{3,10} = G_{8,2}G_{9,3}G_{10,1} \tag{2.26}$$

$$G_{1,8}G_{2,6}G_{4,5} = G_{8,2}G_{6,4}G_{5,1} \tag{2.27}$$

$$G_{3,10}G_{1,5}G_{4,7} = G_{10,1}G_{5,4}G_{7,3} \tag{2.28}$$

$$G_{2,6}G_{4,7}G_{3,9} = G_{6,3}G_{7,3}G_{9,2} \tag{2.29}$$

which are equivalent to (2.2)-(2.5) respectively due to (2.24). We mention that symmetrical equations (2.2)-(2.5) are not independent of one another. In fact, if we assume each $F_{p,q}$ is nonzero, then each of (2.2)-(2.5) may be derived from the other three, for example, we reorder (2.2)-(2.4) as follows

$$\begin{aligned} F_{1,8}F_{2,9}F_{3,10} &= F_{8,2}F_{9,3}F_{10,1}, \\ F_{8,2}F_{6,4}F_{5,1} &= F_{1,8}F_{2,6}F_{4,5}, \\ F_{10,1}F_{5,4}F_{7,3} &= F_{3,10}F_{1,5}F_{4,7}. \end{aligned}$$

Multiplying the first equation by the second and the third ones, we get

$$F_{2,9}F_{6,4}F_{5,1}F_{5,4}F_{7,3} = F_{9,3}F_{2,6}F_{4,5}F_{1,5}F_{4,7}.$$

Noting that $F_{p,q} = -F_{q,p}$, one immediately gets (2,5).

We know that the necessary and sufficient condition to make $R(x,y,z)$ be nonsingular in Ω is to let the denominator of $R(x,y,z)$ take values with the same sign at four vertices of the tetrahedron Ω, i.e., C_1, C_2, C_3 and C_4 must possess the same sign. From this and (2.25) follow immediately (2.6)-(2.11).

We point out that (2.12) and (2.13) are the necessary and sufficient conditions for $R^k(x,y,z)$ and $R^{k+1}(x,y,z)$ to have a C^0-smooth connection over the common boundary of Ω_k and Ω_{k+1}. Let

$$R^k(x,y,z) - R^{k+1}(x,y,z) = \frac{N_k(x,y,z)}{D_k(x,y,z)}, \tag{2.30}$$

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then it follows from (2.12), (2.13) and our convention about the labeling,

$$N_k(1^k) = N_k(3^k) = N_k(4^k) = N_k(5^k) = N_k(7^k) = N_k(10^k) = 0, \tag{2.31}$$

which means by Bezout's theorem ([2]) that the quadratic polynomial $N_k(x, y, z)$ contains such a linear factor $l_k(x, y, z)$ that its locus

$$L_k(x, y, z) = \{(x, y, z) \mid l_k(x, y, z) = 0\} \tag{2.32}$$

covers the common boundary of Ω_k and Ω_{k+1}. In other words, (2.31) implies the C^0-smooth connection of $R^k(x, y, z)$ and $R^{k+1}(x, y, z)$, therefore (2.12) and (2.13) are sufficient while their necessity is obvious. Theorem 1 is hereby completely proved.

Remark 1 Especially if 5, 6, 7, 8, 9 and 10 are midpoints on corresponding edges of tetrahedron Ω, then conditions (2.2)-(2.5) can be written as follows

$$(\alpha_1 - \alpha_8)(\alpha_2 - \alpha_9)(\alpha_3 - \alpha_{10}) = (\alpha_8 - \alpha_2)(\alpha_9 - \alpha_3)(\alpha_{10} - \alpha_1),$$
$$(\alpha_1 - \alpha_8)(\alpha_2 - \alpha_6)(\alpha_4 - \alpha_5) = (\alpha_8 - \alpha_2)(\alpha_6 - \alpha_4)(\alpha_5 - \alpha_1),$$
$$(\alpha_3 - \alpha_{10})(\alpha_1 - \alpha_5)(\alpha_4 - \alpha_7) = (\alpha_{10} - \alpha_1)(\alpha_5 - \alpha_4)(\alpha_7 - \alpha_3),$$
$$(\alpha_2 - \alpha_6)(\alpha_4 - \alpha_7)(\alpha_3 - \alpha_9) = (\alpha_6 - \alpha_4)(\alpha_7 - \alpha_3)(\alpha_9 - \alpha_2).$$

Remark 2 Under conditions (2.2)-(2.4) we can find out a basis system of solutions of equations (2.25)

$$\begin{aligned} C_1 &= CG_{2,9}G_{3,10}G_{4,5}, \quad C_2 = CG_{1,10}G_{3,9}G_{4,5}, \\ C_3 &= CG_{2,9}G_{10,1}G_{4,5}, \quad C_4 = CG_{3,10}G_{2,9}G_{5,1}, \end{aligned} \tag{2.33}$$

where C is an arbitrary nonzero constant.

3. C^1 rational splines

Let

$$H^k_{m,n} = C^k_m C^{k+1}_n (u^k_m v^{k+1}_n - u^{k+1}_n v^k_m), \tag{3.1}$$

and denote by V_k the volume of the tetrahedral cell Ω_k and by $S^k_{p;x}$ the area of the triangle which is the projection of the triangular plane corresponding to vertex p^k onto the YOZ plane.

Theorem 2 *The necessary and sufficient conditions for the existence of* $SR^1_{1,1}(\Omega, V)$ *satisfying* (1.1) *are given by* (2.2)-(2.13) *and*

$$\begin{aligned} V_k \sum_{n=1}^{4} H^k_{1,n} S^{k+1}_{n;t} + V_{k+1} \sum_{m=1}^{4} H^k_{m,1} S^k_{m;t} &= 0, \\ V_k \sum_{n=1}^{4} H^k_{3,n} S^{k+1}_{n;t} + V_{k+1} \sum_{m=1}^{4} H^k_{m,2} S^k_{m;t} &= 0, \\ V_k \sum_{n=1}^{4} H^k_{4,n} S^{k+1}_{n;t} + V_{k+1} \sum_{m=1}^{4} H^k_{m,4} S^k_{m;t} &= 0, \end{aligned} \tag{3.2}$$

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where

$$\frac{\partial D(3, 4, 1, \bullet)}{\partial t} \neq 0, \quad \text{for} \quad t = x, y \text{ or } z. \tag{3.3}$$

Proof From (2.17) and (2.30) we obtain

$$N_k(x, y, z) = \sum_{m=1}^{4}\sum_{n=1}^{4} C_m^k C_n^{k+1} (u_m^k v_n^{k+1} - u_n^{k+1} v_m^k) W_m^k(x, y, z) W_n^{k+1}(x, y, z). \tag{3.4}$$

Since (2.2) - (2.13) are satisfied, we have

$$N_k(x, y, z) = l_k(x, y, z) l_k^*(x, y, z),$$

where $l_k^*(x, y, z)$ is linear. If $L_k(x, y, z)$ is not parallel to, for example, the X-axis, i.e.,

$$\frac{\partial l_k(x, y, z)}{\partial x} \neq 0,$$

then it follows from Bezout's theorem that

$$\frac{\partial N_k(1^k)}{\partial x} = \frac{\partial N_k(3^k)}{\partial x} = \frac{\partial N_k(4^k)}{\partial x} = 0$$

are the necessary and sufficient conditions for $R^k(x, y, z)$ and $R^{k+1}(x, y, z)$ to be C^1 smoothly connected over $\Omega_k \cap \Omega_{k+1}$, which yields (3.2) by means of (2.15), (3.4) and our convention about the labeling at the very beginning that $1^k = 1^{k+1}$, $3^k = 2^{k+1}$ and $4^k = 4^{k+1}$. The theorem is proved.

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三维空间中的有理样条插值

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摘 要

对三维空间某个多面体区域的四面体剖分, 通过在每个四面体胞腔的棱和顶点设置适当的插值结点. 本文给出了(1, 1)型 C^0 及 C^1 光滑的非奇异有理样条存在的充分必要条件.

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