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Infinitely Many Radially Symmetric Solutions to Superlinear Boundary Value Problems in Annular Domains*

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Abstract In this paper we show that a class of superlinear boundary value problems in annular domains have infinitely many radially symmetric solutions. The result is obtained without other restrictions on the growth of the nonlinearities. Our methods rely on the energy analysis and the phase-plane angle analysis of the solutions for the associated ordinary differential equations.

Keywords boundary value problems, radial solutions, shooting methods, annular domains

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1. Introduction

In this short paper we consider the existence of infinitely many radial solutions of the equation

$$\Delta u + f(u) = 0 \quad \text{in} \quad 0 < R_1 < |x| < R_0,\ x \in \mathbf{R}^N,\ N \geq 3, \tag{1.1}$$

with the Dirichlet boundary condition

$$u = 0 \quad \text{on} \quad |x| = R_1 \quad \text{and} \quad |x| = R_0. \tag{1.2}$$

We assume that

$$f \in C^1(-\infty, \infty),\ f(0) = 0. \tag{1.3}$$

This condition on f will be assumed throughout the paper without further mention. In addition, the following conditions will be assumed:

(H$_1$) f is nondecreasing in $(-\infty, \infty)$;

(H$_2$) $\lim_{|t|\to\infty} f(t)/t = \infty$.

Over the last two decades, considerable progress has been made in the study of superlinear boundary value problems, such as:

$$\Delta u + g(u) = q(x), \quad \text{for} \quad x \in \Omega;\ u = 0 \quad \text{for} \quad x \in \partial\Omega, \tag{1.4}$$

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where Ω is a bounded region in $\mathbf{R}^N$, $g: \mathbf{R} \to \mathbf{R}$ is a continuous function, $q \in L^2(\Omega)$ and

$$\lim_{|u| \to \infty} (g(u)/u) = \infty. \tag{1.5}$$

The main goal has been to identify the conditions on Ω, g, q under which (1.4)-(1.5) has infinitely many solutions. Most of the results have been obtained by using variational methods with the growth of g less than the Sobolev inequality growth, see, for example, [2, 3] and [12-14]. Recently, Castro and Kurepa[6] generalized their results to the case that g can surpasses the Sobolev inequality growth and found infinitely many radial solutions of (1.4) with Ω being a ball. Meanwhile, the existence of positive radial solutions of (1.1)-(1.2) has also been studied by many authors. See, for example, [4], [7-10] and the reference therein. When f is superlinear (i.e., it satisfies condition (H_2)) the existence of solutions has been proved under various sets of assumptions, always including a restriction on the growth of f at infinity (see [1, 5-6]). It is known that such a growth condition is, in general, necessary for starlike domains [11]. In the case of the annulus, such a growth conditions is not necessary. In this short paper we show that for f satisfying (H_1), (H_2), there exist infinitely many radial solutions of (1.1)-(1.2). Note that there is no restriction on the growth of f here.

When the domain is a ball or an annulus, one may consider in particular radially symmetric solutions, in which case the problems mentioned above reduce to problems in o.d.e., Thus, in terms of the variable

$$\xi = [(N-2) r^{N-2}]^{-1}. \tag{1.6}$$

Eq. (1.1) obtains the form

$$u''(\xi) + \rho(\xi) f(u(\xi)) = 0, \quad \xi_0 < \xi < \xi_1, \tag{1.1$'$}$$

where

$$\rho(\xi) = [(N-2)\xi]^{-k}, \quad k = \frac{2N-2}{N-2}, \tag{1.7}$$

$$\xi_i = [(N-2) R_i^{N-2}]^{-1}, \quad i = 0, 1, \tag{1.8}$$

the boundary condition becomes

$$u(\xi_0) = u(\xi_1) = 0. \tag{1.2$'$}$$

2 Main results

In this section we shall obtain our main result.

Theorem 2.1 *Suppose f satisfies the conditions (H_1) and (H_2). Then* (1.1)-(1.2) *has infinitely many radially symmetric solutions with* $\max_{(R_1, R_0)} u > 0$.

To prove this theorem, we shall first consider the following initial value problem

$$(r^{N-1} u')' + r^{N-1} f(u) = 0, \quad r > R_1, \tag{2.1}$$

$$u(R_1) = 0,\ u'(R_1) = R_1^{-(N-1)} b, \quad \text{where } b > 0. \tag{2.2}$$

Lemma 2.2 *Assume that f satisfies the conditions (H_1) and (H_2). Then, for any $b>0$, problem (2.1)-(2.2) has a unique solution $u(\cdot, b)$ whose domain of definition is (R_1, ∞).*

Proof It is known from the theory of o.d.e., that there exists a unique local solution of (2.1) near $r=R_1$. Suppose that for some increasing sequence $r_n \in (R_1, \infty)$, $r_n \to \bar{r} \in (R_1, \infty)$ we have

$$\lim_{n\to\infty} (u^2(r_n, b) + (u'(r_n, b))^2) \to \infty.$$

If $(u'(r_n, b))^2$ does not tend to infinity, then by the mean value theorem a new increasing sequence $r_n' \to \bar{r}$ can be found so that $(u'(r_n', b))^2 \to \infty$. Thus, without loss of generality, we can assume $(u'(r_n, b))^2 \to \infty$. Let $F(s) = \int_0^s f(t)\,dt$ and $\tilde{E}(r, b) = (u'(r, b))^2/2 + F(u(r, b))$. Since (H_1) and (H_2) hold, we have $F \geq 0$. Hence

$$\lim_{n\to\infty} \tilde{E}(r_n, b) = \infty. \tag{2.3}$$

On the other hand we have

$$\tilde{E}'(r, b) = -\frac{(N-1)}{r}(u'(r, b))^2 \leq 0.$$

Hence

$$\tilde{E}(r, b) \leq \tilde{E}(R_1, b) = R_1^{-2(N-1)} b^2/2,$$

which contradicts (2.3) and thus the lemma is proved.

Now we consider another form of (2.1)-(2.2):

$$u'' + \rho f(u) = 0 \quad \text{for} \quad \xi < \xi_1, \tag{2.4}$$

$$u(\xi_1) = 0,\ u'(\xi_1) = -b. \tag{2.5}$$

For any $b>0$, problem (2.4)-(2.5) has a unique solution $u(\cdot, b)$ whose maximal domain of definition in $(0, \xi_1)$ will be denoted by $(\bar{\xi}(b), \xi_1)$. A function $u(\xi)$ is a solution of (2.4)-(2.5) if and only if it satisfies the integral equation

$$u(\xi) = b(\xi_1 - \xi) - \int_\xi^{\xi_1} (t - \xi)\rho f(u(t))\,dt, \quad \xi \leq \xi_1. \tag{2.6}$$

From (2.6) it is clear that if u is a positive solution in some interval (α, ξ_1) with $\alpha > \tilde{\xi}(b)$, then

$$u(\xi) \leq b(\xi_1 - \xi) \quad \text{in} \quad (\alpha, \xi_1). \tag{2.7}$$

Therefore if $\alpha > \tilde{\xi}(b)$ the above solution can be extended to the left of α. Denote

$$\xi_0(b) = \inf\{\xi_0 > \tilde{\xi}(b):\ u(\xi, b) > 0 \quad \text{in} \quad (\xi_0, \xi_1)\}.$$

It follows from Lemma 2.2 that $\tilde{\xi}(b) = 0$ if f satisfies (H_1) and (H_2).

The following two lemmas are known from [4] and [10].

Lemma 2.3 *For every $b>0$ there exists a unique point $\tau(b)\in(\xi_0(b),\xi_1)$ at which u attains its maximum $u_m(b)$ over this interval. The function $b\to\tau(b)$ is continuously differentiable in $(0,\infty)$.*

Lemma 2.4 *Let f satisfy (H_1), (H_2). Then the domain of definition of $u(\cdot,b)$ is $(0,\xi_1)$ and $\lim_{b\to\infty}\tau(b)=\xi_1$, $\lim_{b\to\infty}\xi_0(b)=\xi_1$. Furthermore, $\lim_{b\to\infty}u_m(b)=\infty$.*

Define $E(\xi,b)=(u'(\xi,b))^2/2+\rho(\xi)F(u(\xi,b))$, $F(t)=\int_0^t f(s)ds$. We shall obtain the following lemma.

Lemma 2.5 *Let f satisfy (H_1), (H_2). Then*

$$\lim_{b\to\infty}E(\xi,b)=\infty$$

uniformly for $\xi\in[\xi_0,\xi_1]$.

Proof Since $\frac{d}{d\xi}E(\xi,b)=\rho'(\xi)F(u(\xi,b))\le 0$ for $\xi\in(\xi_0,\tau(b))$, then

$$E(\xi,b)\ \ge\ E(\tau(b),b)\ =\ \rho(\tau(b))F(u_m(b)).$$

By the facts that $\lim_{b\to\infty}\rho(\tau(b))=\rho(\xi_1)$ and $\lim_{b\to\infty}F(u_m(b))=\infty$, we have that

$$\lim_{b\to\infty}E(\xi,b)=\infty$$

for all $\xi\in[\xi_0,\xi_1]$.

The arguments above imply that for $b>0$ sufficiently large, there exists a unique solution $u(r,b)$ for the problem

$$(r^{N-1}u')'+r^{N-1}f(u)=0;\quad u(R_1)=0,\ u'(R_1)=b$$

such that there exists $\tilde{\tau}(b)>R_1$ with $u'(\tilde{\tau}(b))=0$ and $\lim_{b\to\infty}\tilde{\tau}(b)=R_1$. Moreover,

$$\lim_{b\to\infty}[(u'(r,b))^2/2+F(u(r,b))]=\infty$$

for all $r\in[R_1,R_0]$.

Proof of Theorem 2.1 The main idea of the proof is similar to that in the proof of Theorem A of [6]. We only give the main differences here. Let $(u(r,d),u'(r,d))\ne(0,0)$ for all $r\in(\tilde{\tau}(b),\bar{r})$. By defining $t^2(r,b)=u^2(r,b)+(u'(r,b))^2$ we see that for $b>0$ there exists a unique continuous argument function $\theta(r,b)$, $r\in[R_1,\bar{r})$, such that

$$u(r,b)=t(r,b)\cos\theta(r,b),\quad u'(r,b)=-t(r,b)\sin\theta(r,b),\quad \theta(\tilde{\tau}(b),b)=0.$$

Note that for $r\in(R_1,\tilde{\tau}(b))$, $-\frac{\pi}{2}<\theta(r,b)<0$ and $\theta(R_1,b)=-\frac{\pi}{2}$. An elementary calculation shows that

$$\theta'(r,b)=\sin^2\theta(r,b)+\frac{\left(f(u(r,b))+\frac{n}{r}u'(r,b)\right)\cos\theta(r,b)}{t(r,b)}.$$

By the continuous dependence of the solutions on the initial conditions we only need to prove that

$$\lim_{b\to\infty}\theta(R_0, b) = \infty. \tag{2.8}$$

(Since $\tilde{\tau}(b)\to R_1$ as $b\to\infty$ and $\theta(\tilde{\tau}(b), b)=0$, then $\theta(R_0, b)>0$.) Let the number T in the proof of Theorem A in [6] be $R_1+(R_0-R_1)/4$ here. A little modification of the proof of Theorem A in [6] implies that (2.8) holds. The proof is complete.

Remark 1 We can obtain the similar results by considering (2.1) with the initial value conditions $u(R_1)=0$ and $u'(R_1)=-R_1^{-(N-1)}b$, $b>0$.

2. We can also obtain infinitely many radial solutions of the equation (1.1) subject to one of the following boundary conditions:

(i) $\frac{du}{dr}=0$ on $|x|=R_1$ and $u=0$ on $|x|=R_0$,

(ii) $u=0$ on $|x|=R_1$ and $\frac{du}{dr}=0$ on $|x|=R_0$.

For the first case, we consider the problem (2.1) with the initial value condition:

$$u(\xi_1)=b>0, \quad u'(\xi_1)=0.$$

Let $E(\xi, b)$ be as in Lemma 2.5. A little modification of the proof of Lemma 2.5 implies that $\lim_{b\to\infty}E(\xi, b)=\infty$ uniformly for $\xi\in[\xi_0, \xi_1]$. The same arguments as in the proof of Theorem 2.1 then imply that the conclusion is true.

We can not directly use the arguments in the proof of case (i) to prove case (ii). In this case we directly consider the initial value problem

$$u''+\frac{N-1}{r}u'+f(u)=0, \quad r<R_0,$$

$$u(R_0)=b, \quad u'(R_0)=0.$$

Let $E(r, b)=(u'(r, b))^2/2+F(u(r, b))$. Then

$$\frac{dE}{dr}=-\frac{N-1}{r}(u')^2\le 0$$

for $r\in(R_1, R_0)$. Thus, $E(r, b)\ge E(R_0, b)=F(b)$. This implies that $\lim_{b\to\infty}E(r, b)=\infty$ for $r\in[R_1, R_0]$.

Let $\xi=R_0-r$, $\tilde{u}(\xi)=u(r)$. Then, we know that $\tilde{E}(\xi, b):=(\tilde{u}'(\xi, b))^2/2+F(\tilde{u}(\xi, b))\to\infty$ in $[0, R_0-R_1]$ as $b\to\infty$. Define $t^2(\xi, b)=\tilde{u}^2(\xi, b)+(\tilde{u}'(\xi, b))^2$. We see that for $b>0$ there exists a unique continuous argument function $\theta(\xi, b)$, $\xi\in[0, \bar{\xi})$, such that

$$\tilde{u}(\xi, b)=t(\xi, b)\cos\theta(\xi, b), \quad \tilde{u}'(\xi, b)=-t(\xi, b)\sin\theta(\xi, b),$$

$$\theta(0, b)=0.$$

The similar arguments to that in the proof of Theorem A in [6] imply that our conclusion is true.

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环形域上超线性边值问题的无穷多径向对称解

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摘 要

本文揭示了一类超线性边值问题在环域上有无限多的正对称解, 此结果对非线性项的增长性除超线性外无其它限制. 本文主要方法是相关常微分方程解的能量分析和相平面分析.