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Some Results on Dual Cosine Operator Function*

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Abstract Some basic properties of dual cosine operator function are given. The concept and characterization of θ-reflexivity with respect to cosine operator function are first studied.

Keywords cosine operator function, duality, θ-reflexivity.

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1. Introduction

Let X be a Banach space and $B(X)$ the algebra of all bounded linear operator on X. A one-parameter family $c(t)$, $t\in R$, in $B(X)$ satisfying $c(o)=I$ and $c(t+s)+c(t-s)=2c(t)c(s)$ is called a cosine operator function on X. If for each $x\in X$, $c(t)x$ is strongly continuous in t, $c(t)$ is called a strongly continuous cosine operator function on X. The associated sine function $s(t)$ is defined as $s(t)x=\int c(s)x\,dx$, $x\in X$, $t\in R$. Strongly continuous cosine operator functions has been studied e.g. in [2][4-9]. Let X^* denotes the dual space of X, then the cosine operator function $c^*(t)=(c(t))^*$ on X^* is called its adjoint. The duality theory for strongly continuous cosine operator function was initiated by Nagy in [5]. One of the difficulties in dealing with dual cosine function is that the adjoint cosine function $c^*(t)$ of strongly continuous cosine function $c(t)$ in a Banach space X needed not be strongly continuous in X^*, unless X is reflexive. An adjoint consine function is weak *-continuous and is weak *-generated by A^*, the adjoint of the generator A of $c(t)$. Some results about weak *-continuous consine operator function has been presented by [7].

In recent years, the duality theory, in particular the perturbation theory, for c_0-semigroup has been extensively developed. However, it seems that there has been no attempt to consider the parallel theory for cosine operator function. In this and following paper, we take an attempt to give some perturbation and duality result for cosine function which is analogous to the theory for c_0-semigroup in [1]. In this paper we restrict ourselves to duality theory. In section 2 we collect some basic properties about duality theory. A part result can be found in [5] and [7]. We introduce the concept of sun-reflexivity (or θ-reflexivity) with

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respect to $c(t)$, which is similar to [1] and some characterizations of θ-reflexivity are given in section 3.

In this paper, elements of X, X^*, X^θ etc. are denoted by x, x^*, x^θ, etc. We use $\langle x, x^*\rangle$ and $\langle x^*, x\rangle$ interchangeably to denote $x^*(x)$, i.e. the value of x^* at x, whenever $x\in X$ and $x^*\in X^*$.

2 Basic property

Let X be a Banach space, and $c(t)$ a strongly continuous cosine operator function, let $c^*(t)$ denote the cosine function of adjoint operator on the dual space and let A^* denote the adjoint of A. It is clear that $c^*(t)$, $t\in R$, satisfies $c^*(0)=I$, $c^*(s+t)+c^*(s-t)=2c^*(s)c^*(t)$, $\|c^*(t)\|=\|c(t)\|$ and $c^*(t)$ is continuous on R with respect to the weak * operator topology of $B(X^*)$[5]. However it may happen that $c^*(t)$ is not a strongly continuous operator function. For example, take $X=L_p(R)$, $1\le p<\infty$ and $(c(t)f)(x)=\frac{1}{2}[f(x+t)+f(x-t)]$, $x, t\in R$ then $c^*(t)$ is strongly continuous cosine operator function on $(L_p(R))^*$ if $p>1$, but not for $p=1$. In fact, $c^*(t)$ is not strongly continuous (or even strongly measurable) in $L^\infty(R)$, see [2] Ex. 3.4, $c^*(t)$ is not strongly continuous operator function in nonreflexive Banach space in general.

The next Lemma collect some well-known results, proof may be found in [6, 7]. We shall use the notation $s(t)$, defined by $s(t)x=\int c(s)x\,ds$, denoted the associated sine function. Then we have for $x\in X$ and $x^*\in X^*$, $\langle x, s^*(t)x^*\rangle=\int\langle x, c^*(t)x^*\rangle ds$, or equivalently $s^*(t)x^*=\int c^*(s)x^*\,ds$.

Lemma 2.1 (*i*) $t\to c^*(t)x^*$ *is weak *-continuous on* R.

(*ii*) $x^*\in D(A^*)$ *iff* $\|c^*(t)x^*-x^*\|=o(t^2)$ $(t\to 0)$ *iff* $\lim_{t\to 0^+} t^{-2}\|c^*(t)x^*-x^*\|<+\infty$ *i.e.*, A^* *is the weak *-generator of* $c^*(t)$.

(*iii*) $c^*(t)D(A^*)\subset D(A^*)$ *and* $\frac{d^2}{dt^2}c^*(t)x^*=A^*c^*(t)x^*=c^*(t)A^*x^*$ *for* $x^*\in D(A^*)$ *in the weak *-differentiable sense.*

(*iv*) $s^*(t)D(A^*)\subset D(A^*)$ *and* $\frac{d}{dt}c^*(t)x^*=A^*s^*(t)x^*=s^*(t)A^*x^*$ *for* $x^*\in D(A^*)$.

(*v*) $\int\int c^*(\rho)x^*\,d\rho ds=\int(t-s)c^*(s)x^*\,ds\in D(A^*)$ *for every* $x^*\in X^*$, *and*

$$A^*\int(t-s)c^*(s)x^*ds=c^*(t)x^*-x^* \text{ for } x^*\in X^*,$$

$$\int(t-s)c^*(s)A^*x^*ds=c^*(t)x^*-x^* \text{ for } x^*\in D(A^*).$$

When X is nonreflexive, $c^*(t)$ need not be strongly continuous and related to that is the fact that A^* need not be densely defined. Now we define subspace X^θ of X^* on which $c^*(t)$ is strongly continuous.

Definition 2.2 $X^\theta=\{x^*\in X^*: \lim_{t\to 0}c^*(t)x^*=x^*\}$.

Let $c^\theta(t)$ be the restriction of $c^*(t)$ to X^θ then $c^\theta(t)$ is strongly continuous cosine operator function and let A^θ be the generator of $c^\theta(t)$.

Lemma 2.3[5], [7] (*i*) X^θ *is a closed linear subspace of* X^*, *for every* $s\in R$, *we have* $c^*(s)X^\theta\subset X^\theta$,

(ii) $D(A^*) \subset \overline{X^\theta}$ and for $x^* \in D(A^*)$ $2\| \underline{(c^*(t) - I^*)x^*} \| \le t^{-2} \|A^* x^*\| \sup_{0\le s\le t} \|c(s)\|$.

(iii) $X^\theta = \overline{D(A^*)}$, or equivalently, $X^\theta = \overline{R(\lambda, A^*)(X^*)}$ for all $\lambda \in \rho(A)$.

(iv) $D(A^\theta) = \{x^* \in D(A^*) : A^* x^* \in X^\theta\}$ and $x^* \in D(A^\theta)$ implies $A^* x^* = A^\theta x^*$.

(v) $D(A^\theta)$ is weak $*$ dense in X^*.

(vi) If X is reflexive then $X^\theta = X^*$ and $A^\theta = A^*$.

(vii) $\rho(A^\theta) = \rho(A^*) = \rho(A)$ and $R(\lambda, A^\theta) = R(\lambda, A^*)|_{X^\theta}$ for all $\lambda \in \rho(A)$.

3 θ-Reflexivity and characterizations

Next we present some argument which is similar to that for c_0-semigroup in [1], [3], [10].

For $x^\theta \in X^\theta$ we define $\|x^\theta\| = \sup\{|\langle x, x^\theta\rangle| : x \in X, \|x\| \le 1\}$ then $(X^\theta, \|\cdot\|)$ is a Banach space since x^θ is closed in X^*. For $x \in X$, we introduce $\|x\|^1 = \sup\{|\langle x, x^\theta\rangle| : x^\theta \in X^\theta, \|x^\theta\| \le 1\}$ then we have

Lemma 3.1 $\|x\|^1 \le \|x\| \le M \|x\|^1$, $\forall x \in X$ where $M =$ const.

Proof It is clear that $\|x\|^1 = \sup\{|\langle x, x^\theta\rangle| : x^\theta \in X^\theta, \|x^\theta\| \le 1\} \le \|x\|$ for every $x \in X$. By Lemma 2.1 (v) and Lemma 2.3 (ii) $2t^{-2}\iint c^*(\rho)x^* d\rho ds \in D(A^*) \subset X^\theta$ and $\|2t^{-2}\iint c^*(\rho)x^* d\rho ds\| \le M \|x^*\|$ for small $|t|$, where $M > 0$ is a constant. Hence

$$\|x\|^1 \ge \sup\{|\langle x, 2t^{-2}M^{-1}\iint c^*(\rho)x^* d\rho ds\rangle| : x^* \in X^*, \|x^*\| \le 1\}$$

$$= \sup\{|\langle 2t^{-2}M^{-1}\iint c(\rho)x\, d\rho ds, x^*\rangle| : x^* \in X^*, \|x^*\| \le 1\}$$

for every $x \in X$ and $t \in R$. By taking the limit $t \to 0$ we have $\|x\| \le M \|x\|^1$, so we finish the proof.

In other words, $\|\cdot\|^1$ is a norm equivalent with the original norm and when $c(t)$ is a contraction cosine operator function the two norms are actually the same. Let $X^{\theta*}$ be the dual of X^θ and $c^{\theta*}(t)$ be the dual cosine function of $c^\theta(t)$ with weak $*$ generator $A^{\theta*}$. We can now repeat the above procedure and obtain the domain of strongly continuity of $c^{\theta*}(t)$ in $X^{\theta*}$, $X^{\theta\theta} = \overline{D(A^{\theta*})}$, and $c^{\theta\theta}(t)$ is a strongly continuous cosine function in $X^{\theta\theta}$ with generator $A^{\theta\theta}$. Note in particular that $\rho(A^{\theta\theta}) = \rho(A^{\theta*}) = \rho(A^\theta) = \rho(A^*) = \rho(A)$ and $R(\lambda, A^{\theta\theta}) = R(\lambda, A^{\theta*})|_{X^{\theta\theta}}$ for all $\lambda \in \rho(A)$, clearly $X^{\theta\theta} = \{x^{\theta*} \in X^{\theta*} : \lim_{t\to 0} c^{\theta*}(t)x^{\theta*} = x^{\theta*}\}$ and $X \subset X^{\theta\theta}$ since $c(t)$ is strongly continuous

Let $j: X \to X^{\theta*}$ be the canonical embedding given by $\langle jx, x^\theta\rangle = \langle x^\theta, x\rangle$, j maps X into $X^{\theta\theta}$. By Lemma 3.1, $j: X \to X^{\theta*}$ is an isometry not necessary surjective and $j(X)$ is closed in $X^{\theta*}$. If X equipped with the norm $\|\cdot\|^1$, then X can be embeded in $X^{\theta\theta}$, in case j maps X onto $X^{\theta\theta}$ we have

Definition 3.2 *X is called θ-reflexive (or sun-reflexive) with respect to $c(t)$ iff $X = X^{\theta\theta}$.*

We have the following characterization of θ-reflexivity:

Theorem 3.3 *X is θ-reflexive with respect to $c(t)$ iff X^θ is θ-reflexive with respect to $c^\theta(t)$.*

Theorem 3.4 *X is θ-reflexive with respect to $c(t)$ iff $R(\lambda, A)$ is weakly compact in the*

X^{θ}-*topology sense*.

The proof of above two theorems are similar to [3], [10] completely.

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余弦算子函数共轭性的一些结果

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摘 要

本文系统的给出余弦算子函数的共轭性理论, 并引入了余弦函数的 θ- 自反概念, 及给出了一些基本特征