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Correction and Analysis of Kornai-Weibull Queue Model with the Waiting Buyers*

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Abstract First of all, this paper shows clearly careless mistake of Kornai-Weibull queue model with the waiting buyers in J. Kornai[4], as well as Kornai-Weibull queue model without waiters[5]. Secondly, strictly prove that revised model has 'normal state', under more natural conditions. We advance theory of Kornai-Weibull queue model such that deeply understand an abuse of the planned economy.

Keywords Kornai-Weibull queue model, waiting buyers, correction, analysis

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1. Kornai-Weibull Queue Model with the Waiting Buyers

J. Kornai and J. W. Weibull(1977), (1978), J. Kornai(1980) and H. Socnowska(1988) establish and research queueing model without waiters in shortage economy and try to give the existence proof of "normal state". Liu Xingquan (1994) have showed clearly careless mistake of Kornai-Weibull queue model, corrected this model and gave bifurcated value S^* by strict logical reasoning, and prove that $S < S^*$ is a necessary and sufficient condition under which revised model has "normal state".

J. Kornai(1980) only gave Kornai-Weibull queue model with the waiting buyers. Up to now, I did not see that who gave existence proof of "normal state" of Kornai-Weibull queue model with the waiting buyers. First of all, this paper shows clearly careless mistake of Kornal-Weibull queue model with the waiting buyers in J. Kornai(1980), as well as Kornai-Weibull queue model without waiters (see Liu Xingquan(1994)). Secondly, strictly prove that revised model has "normal state", under more natural conditions. We advance theory of Kornai-Weibull queue model, such that deeply understand an abuse of the planned economy.

Here, let us redescribe this model. We consider a market trading one good only. The price of the good is constant. This good is being traded in indivisible items. Each buyer can buy only one item. There is only one seller and there are n buyers. There is substitute to be sold in another market. It is assumed that the substitute is cheaper than the good and is

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available without queueing. Average substitute satisfaction time is shorter than the good, that is, the substitute is inferior to the good.

First of all, a buyer must decide whether the price of the good is acceptable for him in the process of shopping. If he doesn't accept the price, he buys the substitue. If he accept the price, he considers the queueing time. If he accept the queueing time, he joins the queue. If he doesn't accept the queueing time, he considers either buys the substitute, or again considers whether joins queue or not after waits a period of time. If he buys the substitute, after substitute satisfaction time, buyer again go into new process of shopping. Queuer has bought the good, after good satisfaction time, he also go into new process of shopping.

So, at any time every buyer is either queueing, or consuming the good, or consuming the substitute, or waiting. At any time, we denote by $y_1(t)$ the number of queueing buyers, by $y_2(t)$ the number of buyers consuming the good, by $y_3(t)$ the number of buyers consuming the substitute, by $y_4(t)$ the number of waiting buyers.

Constant S is called the service capacity, it is the maximal number of buyers served per time unit. We denote by R the service rate, it equal the number of buyers served per time unit. From above relation the process of shopping is described by the following system of equations

$$\begin{cases} \dfrac{dy_1}{dt} = \lambda\varphi(y_1)(\gamma y_2 + xy_3) + \psi\varphi(y_1)y_4 - R, \\ \dfrac{dy_2}{dt} = R - \gamma y_2, \\ \dfrac{dy_3}{dt} = [1 - \lambda + \lambda\mu(1 - \varphi(y_1))](\gamma y_2 + xy_3) + \mu(1 - \varphi(y_1))\psi y_4 - xy_3, \\ \dfrac{dy_4}{dt} = \lambda(1 - \mu)(1 - \varphi(y_1))(\gamma y_2 + xy_3) + (1 - \mu)(1 - \varphi(y_1))\psi y_4 - \psi y_4, \end{cases} \tag{1}$$

where $0 < \lambda \leq 1$ is the initial buying propensity and depends only on the price of the good, $0 \leq \varphi(y_1) \leq 1$ is the queueing propensity and depends on the queue length y_1, $0 < \mu < 1$ is the forced substitute propensity, $\gamma > 0$ is the need-renewal rate of the good, and $x > 0$ is the need-renewal rate of the substitute ($\frac{1}{\gamma}$ is average good satisfaction time, and $\frac{1}{x}$ is average substitute satisfaction time), ψ is the reconsider rate of the waiting buyers ($\frac{1}{\psi}$ is the average waiting time of the buyer). J. Kornai regard.

$$R = \begin{cases} S, & \text{if } y_1 > 0, \\ 0, & \text{if } y_1 = 0. \end{cases}$$

This view is unrealistic. The number of buyers served per time unit depends not only on the queue length y_1, but also on the number of buyers joining the queue in the time unit $\lambda\varphi(y_1)(\gamma y_2 + xy_3) + \psi\varphi(y_1)y_4$. Noting that $y_1 + y_2 + y_3 + y_4 = n$, we have $y_4 = n - y_1 - y_2 - y_3$. We denote $h(y_1, y_2, y_3) = y_1 + \lambda\varphi(y_1)(\gamma y_2 + xy_3) + \psi\varphi(y_1)(n - y_1 - y_2 - y_3)$. Thus

$$R = R(y_1, y_2, y_3) \triangleq \begin{cases} S, & \text{if } h(y_1, y_2, y_3) \geq S; \\ h(y_1, y_2, y_3), & \text{if } h(y_1, y_2, y_3) < S. \end{cases}$$

The revised queueing model with the waiting buyers is

$$\begin{cases} \dfrac{dy_1}{dt} = \lambda\varphi(y_1)(\gamma y_2 + xy_3) + \psi\varphi(y_1)y_4 - R(y_1, y_2, y_3) \\ \dfrac{dy_2}{dt} = R(y_1, y_2, y_3) - \gamma y_2, \\ \dfrac{dy_3}{dt} = [1 - \lambda + \lambda\mu(1 - \varphi(y_1))](\gamma y_2 + xy_3) + \mu(1 - \varphi(y_1))y_4 - xy_3, \\ \dfrac{dy_4}{dt} = \lambda(1 - \mu)(1 - \varphi(y_1))(\gamma y_2 + xy_3) + (1 - \mu)\psi(1 - \varphi(y_1))y_4 - \psi y_4, \end{cases}$$

A set $Y = \{y: y \in R^4, y_j \geq 0 (j = 1, 2, 3, 4), y_1 + y_2 + y_3 + y_4 = n\}$ is called a set of feasible states of market. Putting $y_4 = n - y_1 - y_2 - y_3$, we have

$$\frac{dy}{dt} = f(y), \tag{3}$$

where $y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$,

$$f(y) = \begin{pmatrix} f_1(y_1, y_2, y_3) \\ f_2(y_1, y_2, y_3) \\ f_3(y_1, y_2, y_3) \end{pmatrix} \triangleq \begin{pmatrix} \lambda\varphi(y_1)(\gamma y_2 + xy_3) + \psi\varphi(y_1)(n - y_1 - y_2 - y_3) - R(y_1, y_2, y_3) \\ R(y_1, y_2, y_3) - \gamma y_2, \\ [1 - \lambda + \lambda\mu(1 - \varphi(y_1))](\gamma y_2 + xy_3) \\ \quad + \mu\psi(1 - \varphi(y_1))(n - y_1 - y_2 - y_3) - xy_3 \end{pmatrix}$$

and a set of reduced feasible states $\overline{Y} = \{y: y \in R^3, y_j \geq 0 (j = 1, 2, 3), y_1 + y_2 + y_3 \leq n\}$.

We shall assume that

(I) $\lambda, \gamma, x, \psi, \mu$ and S are constants, and $0 < \lambda \leq 1$, $0 < \gamma < x < \psi$, $0 < \mu < 1$, $S > 0$;

(II) The function $\varphi: [0, \infty] \to [0, 1]$ is nonincreasing, of class C^1, with $\varphi(0) = 1$, $\varphi(y_1) < 1$ $(y_1 > 0)$, $\varphi(y_1) = 0 (y_1 \geq n)$;

(III) $0 < \gamma < x < \frac{1}{2}$;

(IV) $0 \geq \frac{d\varphi(y_1)}{dt} \geq -\frac{1-\lambda\gamma}{\lambda\gamma n}$;

(V) $S < \lambda\gamma n$.

Hypotheses (I) and (II) are practical. Hypothesis (III) is easy to bring about so long as we select suitable unit of time t. For hypothesis (IV), we name $\frac{d\varphi(y_1)}{dt}$ the marginal queueing propensity. $\frac{d\varphi(y_1)}{dy_1} \leq 0$ indicate clearly that queueing propensity is nonincreasing when queue length y_1 increase. We let the marginal queueing propensity be larger than or equal to a specific number $(-\frac{1-\lambda\gamma}{\lambda\gamma n})$. Whether has it some arguments? The answer is clear. It is a long-term shortage economy that we inspect. Buyers have been accustomed to queueing when shopping. They can't lightly change their wishes of joining queue to wait for consuming only since one buyer joins or Leaves the above queue. For hypothesis (V), its interpretation is very explicit. The direct origin of shortage is that the supply capacity is limited.

The main result of this paper is the following theorem.

Theorem *If hypotheses* (I), (II), (III), (IV) *and* (V) *hold, then system* (3) *has only unique stationary point* $\bar{y} \in \operatorname{Int}\bar{Y}$ *on the set* $\bar{Y}$, *and is globally asymptotically stable on the set* $\bar{Y}$.

Let $\bar{Z} = (\bar{y}_1, \bar{y}_2, \bar{y}_3, n - \bar{y}_1 - \bar{y}_2 - \bar{y}_3)^T$, then the set Y of feasible states of system (2) is 3-dimensional stable manifold of the stationary point $\bar{Z}$, and $\bar{y}_1 > 0, \bar{y}_2 > 0, \bar{y}_3 > 0, n - \bar{y}_1 - \bar{y}_2 - \bar{y}_3 > 0$. Now, the stationary point $\bar{Z}$ is called the "normal state" of a market in shortage economy. According to the theorem, if hypotheses (I), (II), (III), (IV) and (V) hold, then a market in shortage economy have the "normal state".

2 Proof of Theroem

By hypotheses (I) and (II), we have

$$\frac{\partial}{\partial y_3} h(y_1, y_2, y_3) = -\varphi(y_1)(\psi - \lambda x) \leq 0,$$

thus

$$h(y_1, y_2, y_3) \geq h(y_1, y_2, n - y_1 - y_2) = y_1 + \lambda\varphi(y_1)(\gamma y_2 + x(n - y_1 - y_2)) \triangleq H(y_1, y_2),$$

for any $(y_1, y_2, y_3)^T \in \bar{Y}$. Noting that

$$\frac{\partial}{\partial y_2} H(y_1, y_2) = -\lambda\varphi(y_1)(x - \gamma) \leq 0,$$

we know

$$H(y_1, y_2) \geq H(y_1, n - y_1) = y_1 + \lambda\varphi(y_1)\gamma(n - y_1) \triangleq \theta(y_1),$$

for any $(y_1, y_2, n - y_1 - y_2) \in \bar{Y}$. From hypothesis (IV), we obtain

$$\frac{d\theta(y_1)}{dy_1} = 1 - \lambda\gamma\varphi(y_1) + \lambda\gamma(n - y_1)\frac{d\varphi(y_1)}{dy_1} \geq 1 - \lambda\gamma - \lambda\gamma_n \frac{1 - \lambda\gamma}{\lambda\gamma n} = 0.$$

Thus $\theta(y_1) \geq \theta(0) = \lambda\gamma n$, for any $0 \leq y_1 \leq n$. In words, for any $(y_1, y_2, y_3)^T \in \bar{Y}$, from hypothesis (V), we obtain

$$h(y_1, y_2, y_3) \geq h(y_1, y_2, n - y_1 - y_2) \geq h(y_1, n - y_1, 0) \geq h(0, n, 0) = \lambda\gamma n > S.$$

This shows, for any $(y_1, y_2, y_3) \in \bar{Y}$, we have $R(y_1, y_2, y_3) \equiv S$.

Thus right side of the system (3) become

$$f(y) = \begin{pmatrix} \lambda\varphi(y_1)(\gamma y_2 + x y_3) + \psi\varphi(y_1)(n - y_1 - y_2 - y_3) - S \\ S - \gamma y_2 \\ [1 - \lambda + \lambda\mu(1 - \varphi(y_1))](\gamma y_2 + x y_3) \\ \quad + \mu\psi(1 - \varphi(y_1))(n - y_1 - y_2 - y_3) - x y_3 \end{pmatrix} \tag{4}$$

Proposition 1 *System* (3) *has exactly one stationary point* $\bar{y} \in \operatorname{Int}\bar{Y}$ *on set* $\bar{Y}$.

Proof In order to find out the stationary point of system (3), we may solute equations

$$\begin{cases} \lambda\varphi(y_1)(\gamma y_2 + xy_3) + \psi\varphi(y_1)(n - y_1 - y_2 - y_3) - S = 0, \\ S - \gamma y_2 = 0, \\ [1 - \lambda + \lambda\mu(1 - \varphi(y_1))](\gamma y_2 + xy_3) \\ \quad + \mu\psi(1 - \varphi(y_1))(n - y_1 - y_2 - y_3) - xy_3 = 0. \end{cases}$$

From second equation of system (5), we obtain $y_2 = \frac{S}{\gamma}\ (0 < \frac{S}{\gamma} < n)$, and to substitute $\frac{S}{\gamma}$ for y_2 in other two equations of system (5), we obtain

$$\begin{cases} \lambda\varphi(y_1)(S + xy_3) + \psi\varphi(y_1)(n - y_1 - \frac{S}{\gamma} - y_3) - S = 0, \\ [1 - \lambda + \lambda\mu(1 - \varphi(y_1))](S + xy_3) + \mu\psi(1 - \varphi(y_1))(n - y_1 - \frac{S}{\gamma} - y_3) - xy_3 = 0. \end{cases} \tag{6}$$

By now, in order to prove proposition 1, we only need to demonstrate that (6) has exactly one solution $(\overline{y_1}, \overline{y_3}) \in \mathrm{Int}\, E$ on set E, where

$$E = \{(y_1, y_3): y_1 \geq 0,\ y_3 \geq 0,\ y_1 + y_3 \leq n - \frac{S}{\gamma}\}.$$

From hypothesis (II), we know that there exists $y_1^* \in (0, n]$, such that $\varphi(y_1) > 0\ (0 \leq y_1 < y_1^*)$ and $\varphi(y_1) = 0\ (y_1^* \leq y_1 \leq n)$. By first equation of system (6), we know that point $(y_1, y_3) \in E$ is not solution of system (6) for $\varphi(y_1) = 0$. Thus, $y_1 \overline{\in} [y_1^*, n]$. Let $y_1^{**} = \min\{y_1^*, n - \frac{S}{\gamma}\}$, $E^* = \{(y_1, y_3): 0 \leq y_3 \leq n - \frac{S}{\gamma} - y_1 (0 \leq y_1 < y_1^{**})\}$.

Up to now, proving proposition 1 is equal to proving that the system (6) has exactly one solution $(\overline{y_1}, \overline{y_3}) \in \mathrm{Int} E^*$ on E^*. On E^*, from first equation of system (6), we have

$$y_3 = \frac{\psi\varphi(y_1)(n - y_1 - \frac{S}{\gamma}) - S + \lambda\varphi(y_1)S}{(\psi - \lambda\chi)\varphi(y_1)}.$$

To substitute the right side of (7) for y_3 of second equation of system (6), we obtain

$$h(y_1) \triangleq [1 - \lambda + \lambda\mu(1 - \varphi(y_1))]S + \mu\psi(1 - \varphi(y_1))(n - y_1 - \frac{S}{\gamma})$$

$$+ [\lambda\chi + \mu(\psi - \lambda\chi)(1 - \varphi(y_1))]\frac{(1 - \lambda\varphi(y_1))S - \psi\varphi(y_1)(n - y_1 - \frac{S}{\gamma})}{(\psi - \lambda\chi)\varphi(y_1)} = 0.$$

We prove that equation $h(y_1) = 0$ has solution $\overline{y_1} \in (0, y_1^{**})$ on $[0, y_1^{**}]$. First of all,

$$h(0) = (1 - \lambda)S + \lambda\chi\frac{(1 - \lambda)S - (n - \frac{S}{\gamma})}{\psi - \lambda\chi}$$

$$= \frac{\psi(1 - \lambda + \frac{\lambda\chi}{\gamma})S - \lambda\chi\psi n}{\psi - \lambda\chi} < \frac{\psi(1 - \lambda + \frac{\lambda\chi}{\gamma})\lambda\gamma n - \lambda\chi\psi n}{\psi - \lambda\chi}$$

$$< -\frac{(1 - \lambda)(\chi - \gamma)\lambda\psi n}{\psi - \lambda\chi} \leq 0,$$

that is $h(0) < 0$. Secondly, when $y_1^* > n - \frac{S}{\gamma}$, that $y_1^{**} = n - \frac{S}{\gamma}$, we have

$$h(y_1^{**}) = [1 - \lambda + \lambda\mu(1 - \varphi(n - \frac{S}{\gamma}))]S$$

$$+ [\lambda X + \mu(\Psi - \lambda X)(1 - \varphi(n - \frac{S}{\gamma}))] \frac{(1 - \lambda\varphi(n - \frac{S}{\gamma}))S}{(\Psi - \lambda X)\varphi(n - \frac{S}{\gamma})} > 0,$$

when $y_1^* \leq n - \frac{S}{\gamma}$, that is $y_1^{**} = y_1^*$, we have $\lim_{y_1 \to y_1^* - 0} h(y_1) = \infty$.

From $h(y_1)$ is continuous on $[0, y_1^{**})$, we know equation $h(y_1) = 0$ has solution $\overline{y_1} \in (0, y_1^{**})$ on $[0, y_1^{**}]$. To substitute by $\overline{y_1}$ for y_1 of the right side of (7), we obtain $\overline{y_3}$. From $h(\overline{y_1}) = 0$, we can obtain that

$$[1 - \lambda + \lambda\mu(1 - \varphi(\overline{y_1}))]S + \mu\Psi(1 - \varphi(\overline{y_1}))(n - \overline{y_1} - \frac{S}{\gamma})$$

$$= [\lambda X + \mu(\Psi - \lambda X)(1 - \varphi(\overline{y_1}))]\overline{y_3}.$$

From $[1 - \lambda + \lambda\mu(1 - \varphi(\overline{y_1}))]S > 0$, $\mu\Psi(1 - \varphi(\overline{y_1}))(n - \overline{y_1} - \frac{S}{\gamma}) > 0$ and $[\lambda X + \mu(\Psi - \lambda X)(1 - \varphi(\overline{y_1}))] > 0$, we know $\overline{y_3} > 0$. To substitute $\overline{y_1}$ and $\overline{y_3}$ for y_1 and y_3 of system (6) respectively, and first equality of system (6) added to second equality, we obtain

$$\Psi[\varphi(\overline{y_1}) + \mu(1 - \varphi(\overline{y_1}))](n - \overline{y_1} - \frac{S}{\gamma} - \overline{y_3}) = \lambda(1 - \mu)(1 - \varphi(\overline{y_1}))(S + x\overline{y_3}).$$

From $\lambda(1 - \mu)(1 - \varphi(\overline{y_1}))(S + x\overline{y_3}) > 0$, $\Psi[\varphi(\overline{y_1}) + \mu(1 - \varphi(\overline{y_1}))] > 0$, we have

$$n - \overline{y_1} - \frac{S}{\gamma} - \overline{y_3} > 0.$$

Thus $0 < \overline{y_3} < n - \frac{S}{\gamma} - \overline{y_1}$, $0 < \overline{y_1} < y^{**} \leq n - \frac{S}{\gamma}$, this mean $(\overline{y_1}, \overline{y_3}) \in \mathrm{Int}\, E^*$. In order to prove system (6) has exactly one solution $(\overline{y_1}, \overline{y_3}) \in \mathrm{Int} E$ on E, we only need to demonstrate that the equation $h(y_1) = 0$ has exactly one solution $\overline{y_1}$ on $[0, y_1^{**}]$. For this purpose, we only need demonstrate that for any $\overline{y_1} \in (0, y_1^{**})$, if $h(\overline{y_1}) = 0$, then $\frac{d}{dy_1} h(\overline{y_1}) > 0$. In fact,

$$\left.\frac{dh(y)}{dy_1}\right|_{y_1 = \overline{y_1}} = -\lambda\mu\dot{\varphi}(\overline{y_1})S - \mu\Psi(1 - \varphi(\overline{y_1})) - \mu\Psi\dot{\varphi}(\overline{y_1})(n - \overline{y_1} - \frac{S}{\gamma})$$

$$+ \mu(\Psi - \lambda X)\dot{\varphi}(\overline{y_1}) \frac{\Psi\varphi(\overline{y_1})(n - \overline{y_1} - \frac{S}{\gamma}) - (1 - \lambda\varphi(\overline{y_1}))S}{(\Psi - \lambda X)\varphi(\overline{y_1})}$$

$$- [\lambda X + \mu(\Psi - \lambda X)(1 - \varphi(\overline{y_1}))] \cdot \frac{1}{(\Psi - \lambda X)\varphi^2(\overline{y_1})}$$

$$\cdot \{[\Psi\dot{\varphi}(\overline{y_1})(n - \overline{y_1} - \frac{S}{\gamma}) - \Psi\varphi(\overline{y_1}) + \lambda\dot{\varphi}(\overline{y_1})S]\varphi(\overline{y_1})$$

$$- [\Psi\varphi(\overline{y_1})(n - \overline{y_1} - \frac{S}{\gamma}) - (1 - \lambda\varphi(\overline{y_1}))S]\dot{\varphi}(\overline{y_1})\}$$

Noting that

$$\bar{y}_3 = \frac{\psi\varphi(\bar{y}_1)\left(n - \bar{y}_1 - \frac{S}{\gamma}\right) - (1 - \lambda\varphi(\bar{y}_1))S}{(\psi - \lambda\chi)\varphi(\bar{y}_1)},$$

we have

$$\frac{dh(\bar{y}_1)}{dy_1} = -\lambda\mu\dot{\varphi}(\bar{y}_1)(S + x\bar{y}_3) - \mu\psi\dot{\varphi}(\bar{y}_1)\left(n - \bar{y}_1 - \frac{S}{\gamma} - \bar{y}_3\right)$$

$$- \mu\psi(1 - \varphi(\bar{y}_1)) - [\lambda\chi + \mu(\psi - \lambda\chi)(1 - \varphi(\bar{y}_1))]\frac{-\psi\varphi^2(\bar{y}_1) + \dot{\varphi}(\bar{y}_1)S}{(\psi - \lambda\chi)\varphi^2(\bar{y}_1)}$$

$$\geq \frac{1}{(\psi - \lambda\chi)\varphi^2(\bar{y}_1)}\{\lambda\chi\psi\varphi^2(\bar{y}_1) - [\lambda\chi + \mu(\psi - \lambda\chi)(1 - \varphi(\bar{y}_1))]S\dot{\varphi}(\bar{y}_1)\}$$

$$\geq \frac{\lambda\chi\psi}{\psi - \lambda\chi} > 0.$$

Proposition 2 *The stationary point $\bar{y}$ of system* (3) *is asymptotically stable.*

Proof We consider

$$A \triangleq \frac{\partial f(y)}{\partial y}\bigg|_{y=\bar{y}} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix},$$

where

$$a_{11} = \frac{\partial f_1}{\partial y_1}\bigg|_{y=\bar{y}} = \lambda\dot{\varphi}(\bar{y}_1)(\gamma\bar{y}_2 + x\bar{y}_3) + \psi\dot{\varphi}(\bar{y}_1)(n - \bar{y}_1 - \bar{y}_2 - \bar{y}_3) - \psi\varphi(\bar{y}_1),$$

$$a_{12} = \frac{\partial f_1}{\partial y_2}\bigg|_{y=\bar{y}} = \lambda\gamma\varphi(\bar{y}_1) - \psi\varphi(\bar{y}_1),$$

$$a_{13} = \frac{\partial f_1}{\partial y_3}\bigg|_{y=\bar{y}} = \lambda\chi\varphi(\bar{y}_1) - \psi\varphi(\bar{y}_1),$$

$$a_{21} = \frac{\partial f_2}{\partial y_1}\bigg|_{y=\bar{y}} = 0,$$

$$a_{22} = \frac{\partial f_2}{\partial y_2}\bigg|_{y=\bar{y}} = -\gamma,$$

$$a_{23} = \frac{\partial f_2}{\partial y_3}\bigg|_{y=\bar{y}} = 0,$$

$$a_{31} = \frac{\partial f_3}{\partial y_1}\bigg|_{y=\bar{y}} = -\lambda\mu\dot{\varphi}(\bar{y}_1)(\gamma\bar{y}_2 + x\bar{y}_3) - \mu\psi\dot{\varphi}(\bar{y}_1)(n - \bar{y}_1 - \bar{y}_2 - \bar{y}_3) - \mu\psi(1 - \varphi(\bar{y}_1)),$$

$$a_{32} = \frac{\partial f_3}{\partial y_2}\bigg|_{y=\bar{y}} = [1 - \lambda + \lambda\mu(1 - \varphi(\bar{y}_1))]\gamma - \mu\psi(1 - \varphi(\bar{y}_1)),$$

$$a_{33} = \frac{\partial f_3}{\partial y_3}\bigg|_{y=\bar{y}} = [1 - \lambda + \lambda\mu(1 - \varphi(\bar{y}_1))]\chi - \mu\psi(1 - \varphi(\bar{y}_1)) - \chi$$

Thus, characteristic polynomial of A is

$$P(\lambda) \triangleq \det(\lambda I - A) = (\lambda - a_{22})[\lambda^2 - (a_{11} + a_{33})\lambda + (a_{11}a_{33} - a_{13}a_{31})].$$

This shows, one characteristic root of A is $\lambda_1 = a_{22} = -\gamma < 0$, and other characteristic roots of A is roots of quadratic equation

$$\lambda^2 - (a_{11} + a_{33})\lambda + (a_{11}a_{33} - a_{13}a_{31}) = 0.$$

Since $a_{11} \leq -\psi\varphi(\overline{y_1}) < 0$, $a_{33} = -\lambda[1 - \mu(1 - \varphi(\overline{y_1}))]x - \mu\psi(1 - \varphi(\overline{y_1})) < 0$, therefore $-(a_{11} + a_{33}) > 0$. Noting that

$$\begin{aligned}
& a_{11}a_{33} - a_{13}a_{31} \\
&= [-\lambda\dot{\varphi}(\overline{y_1})(S + x\overline{y_3}) - \psi\dot{\varphi}(\overline{y_1})(n - \overline{y_1} - \frac{S}{\gamma} - \overline{y_3}) + \psi\varphi(\overline{y_1})] \\
&\quad \cdot [\lambda x + \mu(\psi - \lambda x)(1 - \varphi(\overline{y_1}))] + (\psi - \lambda x)\varphi(\overline{y_1}) \\
&\quad \cdot [-\lambda\mu\dot{\varphi}(\overline{y_1})(S + x\overline{y_3}) - \mu\psi\dot{\varphi}(\overline{y_1})(n - \overline{y_1} - \frac{S}{\gamma} - \overline{y_3}) - \mu\psi(1 - \varphi(\overline{y_1}))] \\
&\geq \psi[\lambda x + \mu(\psi - \lambda x)(1 - \varphi(\overline{y_1}))]\varphi(\overline{y_1}) - \mu\psi(\psi - \lambda x)(1 - \varphi(\overline{y_1}))\varphi(\overline{y_1}) \\
&= \lambda x\psi\varphi(\overline{y_1}) > 0,
\end{aligned}$$

we know $\mathrm{Re}\lambda_j < 0$ $(j = 2, 3)$. In words, the stationary point $\overline{y}$ of system (3) is an asymptotically stable.

Proposition 3 *For every $y_0 \in \overline{Y}$ there exists unique solution $y(t; y_0)$ of system (3) such that $y|_{t=0} = y_0$ and $y(t; y_0)$ is defined for all $t \geq 0$ and $y(t; y_0) \in \overline{Y}$ for all $t \geq 0$.*

Proof According hypothesis (I), we extend the range of definition of $\varphi(y_1)$ to $(-\delta, +\infty)$ $(\delta > 0$, enough small), such that $\varphi(y_1)$ is monotone non-increasing and continuously differentiable on $(-\delta, +\infty)$. Consequently, the right side of system (3) is continuous and satisfy Lipschetz condition on the bounded domain

$$\Omega \triangleq \{y: y \in R^3, -\delta < y_1, y_2, y_3 < n + \delta, y_1 + y_2 + y_3 < n + \delta\}.$$

So, for every $y_0 \in \Omega$, there exists exactly one solution $y(t; y_0)$ of the system (3) on $[0, \beta(y_0))$, where $[0, \beta(y_0))$ is a maximal interval to which the solution extends right in the domain Ω. Up to now, we only need to prove that for every $y_0 \in \overline{Y}$, $\beta(y_0) = +\infty$ and $y(t; y_0) \in \overline{Y}$ $(t \geq 0)$. Noting that $\overline{Y}$ is a bounded and closed set and $\overline{Y} \subset \Omega$, on the basis of chapter I of Hale, Jack K. (1969) (Theorem 2.1), we only need proving that for every $y_0 \in \overline{Y}$, the solution $y(t; y_0) \in \overline{Y}$ $(0 \leq t < \beta(y_0))$.

On plane L_1: $y_1 = 0$ $(0 \leq y_2, 0 \leq y_3, y_2 + y_3 \leq n)$,

$$\begin{aligned}
\frac{dy_1}{dt} &= \lambda(\gamma y_2 + xy_3) + \psi(n - y_2 - y_3) - S \geq \lambda\gamma(y_2 + y_3) + \lambda\gamma(n - y_2 - y_3) - S \\
&\geq \lambda\gamma n - S > 0.
\end{aligned}$$

According to continuity of function $f_1(y_1, y_2, y_3)$, there exists the $\delta_1^{(j)} > 0$, such that $\lim_{j\to\infty}\delta_1^{(j)} = 0$ decreasingly, and on plane $y_1 = \delta_1^{(j)}$ $(0 \leq y_2, 0 \leq y_3, y_2 + y_3 \leq n - \delta_1^{(j)})$, $\frac{dy_1}{dt} > 0$ $(j = 1, 2, \cdots)$. On plane L_2: $y_2 = 0$ $(\delta_1^{(j)} \leq y_1 \leq n, 0 \leq y_3 \leq n - \delta_1^{(j)}, y_1 + y_3 \leq n)$, $\frac{dy_2}{dt} = S > 0$. According to continuity of function $f_2(y_1, y_2, y_3)$, there exists $\delta_2^{(j)} > 0$, such that $\lim_{j\to\infty}\delta_2^{(j)} = 0$

decreasingly, and on plane $y_2=\delta_2^{(j)}$ ($\delta_1^{(j)}\le y_1\le n-\delta_2^{(j)}$, $0\le y_3\le n-\delta_1^{(j)}-\delta_2^{(j)}$, $y_1+y_2\le n-\delta_2^{(j)}$), $\frac{dy_2}{dt}>0$ ($j=1,2,\cdots$). On plane L_3: $y_3=0$ ($\delta_1^{(j)}\le y_1\le n-\delta_2^{(j)}$, $\delta_2^{(j)}\le y_2\le n-\delta_1^{(j)}$, $y_1+y_2\le n$),

$$\frac{dy_3}{dt}=[1-\lambda+\lambda\mu(1-\varphi(y_1))]\gamma y_2+\mu\psi(1-\varphi(y_1))(n-y_1-y_2)$$
$$\ge[1-\lambda+\lambda\mu(1-\varphi(\delta_1^{(j)}))]\gamma\delta_2^{(j)}>0.$$

According to continuity of function $f_3(y_1,y_2,y_3)$, there exists $\delta_3^{(j)}>0$, such that $\lim_{j\to\infty}\delta_3^{(j)}=0$ decreasingly, and on plane $y_3=\delta_3^{(j)}$ ($\delta_1^{(j)}\le y_1\le n-\delta_2^{(j)}-\delta_3^{(j)}$, $\delta_2^{(j)}\le y_2\le n-\delta_1^{(j)}-\delta_3^{(j)}$, $y_1+y_2\le n-\delta_3^{(j)}$), $\frac{dy_3}{dt}>0$. Let

$$\overline{Y_1}=\{y: y\in R^3,\ \delta_1^{(j)}\le y_1,\ \delta_2^{(j)}\le y_2,\ \delta_3^{(j)}\le y_3,\ y_1+y_2+y_3\le n\},$$

then $\partial\overline{Y_j}=L_1^{(j)}\cup L_2^{(j)}\cup L_3^{(j)}\cup L_4^{(j)}$, where

$L_1^{(j)}: y_1=\delta_1^{(j)}$ ($\delta_2^{(j)}\le y_2\le n-\delta_1^{(j)}-\delta_3^{(j)}$, $\delta_3^{(j)}\le y_3\le n-\delta_1^{(j)}-\delta_2^{(j)}$, $y_2+y_3\le n-\delta_1^{(j)}$),

$L_2^{(j)}: y_2=\delta_2^{(j)}$ ($\delta_1^{(j)}\le y_1\le n-\delta_2^{(j)}-\delta_3^{(j)}$, $\delta_3^{(j)}\le y_3\le n-\delta_1^{(j)}-\delta_2^{(j)}$, $y_1+y_3\le n-\delta_2^{(j)}$),

$L_3^{(j)}: y_3=\delta_3^{(j)}$ ($\delta_1^{(j)}\le y_1\le n-\delta_2^{(j)}-\delta_3^{(j)}$, $\delta_2^{(j)}\le y_2\le n-\delta_1^{(j)}-\delta_3^{(j)}$, $y_1+y_2\le n-\delta_3^{(j)}$),

$L_4^{(j)}: y_3=n-y_1-y_2$ ($\delta_1^{(j)}\le y_1\le n-\delta_2^{(j)}-\delta_3^{(j)}$, $\delta_2^{(j)}\le y_2\le n-\delta_1^{(j)}-\delta_3^{(j)}$, $y_1+y_2\le n-\delta_3^{(j)}$).

We know, there is $\frac{dy_1}{dt}>0$ on $L_1^{(j)}$, there is $\frac{dy_2}{dt}>0$ on $L_2^{(j)}$, and there is $\frac{dy_3}{dt}>0$ on $L_3^{(j)}$. On $L_4^{(j)}$, the outer normal vector is $(1,1,1)^T$, then we have

$$(1,1,1)f(y)\Big|_{y\in L_4^{(j)}}=(1,1,1)\begin{pmatrix}\lambda\varphi(y_1)(\gamma y_2+xy_3)-S\\ S-\gamma y_2\\ [1-\lambda+\lambda\mu(1-\varphi(y_1))](\gamma y_2+xy_3)-xy_3\end{pmatrix}$$
$$=-\lambda(1-\mu)(1-\varphi(y_1))(\gamma y_2+xy_3)\le-\lambda(1-\mu)(1-\varphi(\delta_1^{(j)}))(\gamma\delta_2^{(j)}+x\delta_3^{(j)})<0.$$

From this, for every $y_0\in\overline{Y_j}$, solution $y(t;y_0)\in\overline{Y_j}$ $(0\le t<\beta(y_0))$, where $[0,\beta(y_0))$ is a maximal interval to which the solution $y(t;y_0)$ extends right. From $\overline{Y_j}\subset\overline{Y_{j+1}}$ ($j=1,2,\cdots$), $\bigcup_{j=1}^{\infty}\overline{Y_j}=\mathrm{Int}\overline{Y}$ and continuous dependence of the solution $y(t;y_0)$ of system (3) on the initial value y_0, we can demonstrate that for every $y_0\in\overline{Y}$, also $y(t;y_0)\in\overline{Y}$ $(0\le t<\beta(y_0))$.

Proposition 4 *The stationary point $\overline{y}$ of system* (3) *is golbal asymptotically stable on $\overline{Y}$.*

Before all, we prove two lemma.

Lemma 1 *Plane triangle L: $y_2=\frac{S}{\gamma}$ ($0\le y_1$, $0\le y_3$, $y_1+y_3\le n-\frac{S}{\gamma}$) is stable manifold of the stationary point $\overline{y}$, that is for every $y_0\in L$, we have $y(t;y_0)\in L$ $(0\le t<\infty)$, and $\lim_{t\to\infty}y(t;y_0)=\overline{y}$.*

Proof Obviously, $y_2 = \frac{S}{\gamma} = \overline{y_2}$ is constant solution of second equation of system (3), therefore L is orbital manifold of system (3). For every $y_0 \in L$, we have $y_2(t; y_0) = \frac{S}{\gamma}$, and $y_1(t; y_0)$, $y_3(t; y_0)$ satisfy system

$$\begin{cases} \frac{dy_1}{dt} = \lambda\Psi\varphi(y_1)(S + xy_3) + \psi\varphi(y_1)(n - y_1 - \frac{S}{\gamma} - y_3 - S \triangleq \hat{f}_1(y_1, y_3), \\ \frac{dy_3}{dt} = [1 - \lambda + \lambda\mu(1 - \varphi(y_1))](S + xy_3) + \mu\Psi(1 - \varphi(y_1))(n - y_1 - \frac{S}{\gamma} - y_3) \\ - xy_3 \triangleq \hat{f}_2(y_1, y_3). \end{cases} \tag{8}$$

From Proposition 1, we know $\begin{bmatrix} \overline{y_1} \\ \overline{y_3} \end{bmatrix}$ is unique stationary point of system (8) on L. From Proposition 3, in order to prove this lemma, we only need to prove that $\lim_{t\mapsto\infty} \begin{pmatrix} y_1 & (t; y_0) \\ y_3 & (t; y_0) \end{pmatrix} = \begin{bmatrix} \overline{y_1} \\ \overline{y_3} \end{bmatrix}$. Noting that

$$\frac{\partial \hat{f}_1}{\partial y_1} = \lambda\dot{\Psi}(y_1)(S + \chi y_3) + \psi\dot{\varphi}(y_1)(n - y_1 - \frac{S}{\gamma} - y_3) - \psi\varphi(y_1),$$

$$\frac{\partial \hat{f}_2}{\partial y_3} = [1 - \lambda + \lambda\mu(1 - \varphi(y_1))]\chi - \mu\Psi(1 - \varphi(y_1)) - x = -\lambda x - \mu(\Psi - \lambda\chi)(1 - \varphi(y_1)),$$

we obtain

$$\frac{\partial \hat{f}_1}{\partial y_1} + \frac{\partial \hat{f}_2}{\partial y_3} \le -\lambda x < 0, \text{ for } 0 \le y_1,\ 0 \le y_3,\ y_1 + y_3 \le n - \frac{S}{\gamma}.$$

Thus, according to Bendixson discriminant (see G. Sansone, and R. Conti (1964), Chap 4 §2, Theorem 21), we know that system (8) has not closed orbit on L. From Proposition 3, we know positive semi-orbit of system (8) $\begin{pmatrix} y_1 & (t; y_0) \\ y_3 & (t; y_0) \end{pmatrix} \in L$ $(0 \le t < \infty)$. Noting that L is bounded and closed set, thus ω-limit set Ω of semi-orbit $\begin{pmatrix} y_1 & (t; y_0) \\ y_3 & (t; y_0) \end{pmatrix}$ $(0 \le t < \infty)$ is nonempty set and $\Omega \subset L$. Since system (8) has not any closed orbit on L, therefore unique stationary point of system (8) $\begin{pmatrix} \overline{y_1} \\ \overline{y_3} \end{pmatrix} \in \Omega$ on L (see Pioncare-Bendixson theorem). From asymptotically stability of the stationary point $\begin{pmatrix} \overline{y_1} \\ \overline{y_3} \end{pmatrix}$ and Pioncare-Bendixson theorem, we demonstrate that ω-limit set Ω not contain ordinary point. Thus $\Omega = \left\{ \begin{pmatrix} \overline{y_1} \\ \overline{y_3} \end{pmatrix} \right\}$. Obviously, we have $\lim_{t\mapsto\infty} \begin{pmatrix} y_1 & (t; y_0) \\ y_3 & (t; Y_0) \end{pmatrix} = \begin{pmatrix} \overline{y_1} \\ \overline{y_3} \end{pmatrix}$.

Lemma 2 *There exists $\delta > 0$, such that for every $y_0 \in D(L, \delta)$, we have* $\lim_{t\mapsto\infty} y(t; y_0) = \overline{y}$, *where* $D(L, \delta) = \overline{Y} \cap [\cup_{y^* \in L} \{y: |y - y^*| < \delta\}]$.

Proof From asymptotically stability of the stationary point $\bar{y}$, we know that there exists $\delta_0 > 0$, such that for every $y_0 \in B(\bar{y}, \delta_0) \triangleq \{y: |y - \bar{y}| < \delta_0\}$, we have $\lim_{t\mapsto\infty} y(t; y_0) = \bar{y}$. For bounded and closed set $\hat{L} = L / B(\bar{y}, \delta_0)$, to apply the continuous dependence of the solution $y(t; y_0)$ on the initial value y_0, we demonstrate that for every $\hat{y} \in \hat{L}$, there is $\delta(\hat{y}) > 0$ and $T(\hat{y}) > 0$, such that for every $y_0 \in B(\hat{y}, \delta(\hat{y})) \cap \bar{Y}$, we have $y(T(\hat{y}); y_0) \in B(\bar{y}, \delta_0)$, therefore $\lim_{t\mapsto\infty} y(t; y_0) = \bar{y}$. Set $\{B(\hat{y}, \delta(\hat{y}))\}_{\hat{y}\in\hat{L}} \cup \{B(\bar{y}, \delta_0)\}$ is one open covering of the bounded and closed set L. Thus, there exists $\delta > 0$ such that for every $y_0 \in L$, we have

$$B(y_0, \delta) \subset B(\bar{y}, \delta_0) \cup [\bigcup_{\hat{y}\in\hat{L}} B(\hat{y}, \delta(\hat{y}))],$$

therefore

$$D(L, \delta) \subset \{B(\bar{y}, \delta_0) \cup [\bigcup_{\hat{y}\in\hat{L}} B(\hat{y}, \delta(\hat{y}))]\} \cap \bar{Y}.$$

Obviously, for every $y_0 \in D(L, \delta)$, we have $\lim_{t\mapsto\infty} y(t; y_0) = \bar{y}$.

Proof of Proposition 4 From Proposition 2, we know that the stationary point $\bar{y}$ of system (3) is asymptotically stable. For every $y_0 \in \bar{Y}$, according to Proposition 3, we have $y(t; y_0) \in \bar{Y}$ $(0 \le t < \infty)$. If $y_0 \in D(L, \delta)$, according to Lemma 2, we know $\lim_{t\mapsto\infty} y(t; y_0) = \bar{y}$. If $y_0 \bar{\in} D(L, \delta)$, we consider

$$\xi(y(t; y_0), L) = \inf_{\hat{y}\in L}\{|y(t; y_0) - \hat{y}|\}, \text{ for } t \ge 0.$$

Obviously,

$$\xi(y(t; y_0), L) \le \sqrt{2}\,\left|y_2(t; y_0) - \frac{S}{\gamma}\right|.$$

From second equation of system (3), we know

$$y_2(t; y_0) = \left(y_{20} - \frac{S}{\gamma}\right)e^{-\gamma t} + \frac{S}{\gamma},$$

where $(y_{10}, y_{20}, y_{30})^T = y_0$, thus

$$\xi(y(t; y_0), L) \le \sqrt{2}\,\left|y_{20} - \frac{S}{\gamma}\right| e^{-\gamma t}.$$

Therefore, we have $T(y_0) > 0$ such that $\xi(y(T(y_0); y_0), L) < \delta$, it means $y(T(y_0); y_0) \in D(L, \delta)$. This shows $\lim_{t\mapsto\infty} y(t; y_0) = \bar{y}$ (see Lemma 2).

The theorem of Chapter 1 follows from Proposition 1, 2, 3 and 4.

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具有等待者的Kornai-Weibull排队模型的修正与分析

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摘 要

这篇文章首先明确地指出, 具有等待的买者的Kornai-Weibull排队模型与没有等待者的Kornai-Weibull排队模型具有相同的纰漏. 其次, 本文在较自然的条件下严格地证明了修正后的模型存在"正常状态". 研究Kornai-Weibull排队模型理论能够深刻地揭示计划经济的弊端.