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Inner Associative Rings*

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Abstract Influenced by the paper of Zhang, here we define inner associative rings for non-associative rings, study and obtain some properties about these rings.

Keywords associative ring.

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Throughout this paper R denotes a nonassociative ring, i.e., $R = KL$, where K is a commutative ring with unit or a field and L a loop.

Definition 1 *Let R be a non-associative R. R is said to be inner associative if every proper subring of R is associative but R is non-associative.*

In this note we study the inner associative nature of loop rings, i.e., loops over ring. For more about loop rings please refer to [1].

Definition 2 *Let R be a ring. R is said to be quasi-inner associative if R has at least one proper subring which is associative.*

Proposition 1 *Let R be a inner associative ring. Then R is quasi-inner associative.*

Proposition 2 *Every quasi-inner associative ring need not be inner associative.*

Proof By an example. Take $L = \{e, a_1, a_2, \cdots, a_{15}\}$ be a set. Define "·" on L as follows:

$$a_i \cdot a_i = e \text{ for all } a_i \in L \setminus \{e\}, \quad a_i \cdot e = e \cdot a_i = a_i \text{ for all } a_i \in L,$$

$$a_j \cdot a_j = a_t \text{ where } t = \{2 \cdot j - i\} \bmod 15$$

for all $i \neq j$, $i, j \in \{1, 2, \cdots, 15\}$ and $a_i, a_j \in L \setminus \{e\}$. Clearly L is a loop.

Consider $Z_2 = \{0, 1\}$ the prime field of characteristic two and let Z_2L be the loop ring of L over Z_2. Denote $R = Z_2L$. Clearly R has $S = \{0, e + a_1 + \cdots + a_{15}\}$ to be a proper subring of R which is associative. Hence $R = Z_2L$ is a quasi-inner associative ring. But R is not inner associative for take the subloop $M = \{e, a_1, a_4, a_7, a_{10}, a_{13}\}$, M is nonassociative. Consider Z_2M which is clearly a proper subring of $Z_2L = R$. Now Z_2M is a non-associative propersubring of $Z_2L = R$. Now Z_2M is a non-associative propersubring of R so $Z_2L = R$ is not a inner associa-

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tive ring. Hence the claim.

Theorem 3 *Let L be a finite loop. K any field. The loop ring KL is always quasi-inner associative.*

Proof Let $L = \{m_i / i = 1, 2, \cdots, n\}$, i.e., L is a loop of order n. Take K any field; KL the loop ring of the loop L over the field K.

Now $S = \{0, a \sum_{i=1}^{n} m_i / a \in K\}$ is a subring of KL which is a associative ring. Hence the claim.

Theorem 4 *Let L be a power associative (diassociative loop). K any field. The loop ring KL is quasi-inner associative.*

Proof Obvious by the every definition of power associative or diaassociative loops

Let L be an infinite loop; obtain conditions under which the loop ring KL will be:

(i) quasi-inner associative;

(ii) inner associative.

Does there exists finite loop L and K any field; such that the loop ring KL is inner associative? The answer is 'yes' for if we take L to be a loop given by the following table: Let $L = \{e, a_1, a_2, a_3, a_4, a_5\}$.

*	e	a_1	a_2	a_3	a_4	a_5
e	e	a_1	a_2	a_3	a_4	a_5
a_1	a_1	e	a_3	a_5	a_2	a_4
a_2	a_2	a_5	e	a_4	a_1	a_3
a_3	a_3	a_4	a_1	e	a_5	a_2
a_4	a_4	a_3	a_5	a_2	e	a_1
a_5	a_5	a_2	a_4	a_1	a_3	e

$Z_2 = \{0, 1\}$ be the prime field of characteristic two. Z_2L the loop ring of the loop L over Z_2. Z_2L is inner associative easily verified. L is also power associative but not diassociative.

Proposition 5 *Let L be any loop in which $x^2 = e$ for at least one $x \in L \setminus \{e\}$. K any field. The loop ring KL is quasi-inner associative.*

Proof Take $S = \{0, m(1 + x) / m \in K\}$. S is an associative subring of KL. Hence the claim.

Problem Does there exist a loop L such that its loop ring KL for any field K is not quasi inner associative? Characterize those loops L such that its loop ring KL is inner associative.

References

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