Journal of Mathematical Research & Exposition
Vol. 18, No. 2, 225- 227, May 1998

An Inequality of Matrix and Bayes Unbiased Estimates*

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Abstract The inequality of arithmetic mean and harmonic mean is generalized to the positive definite matrix. With this inequality, we get the optimal properties of Bayes unbiased estimates.

Keywords positive definite matrix, conditional expectation, inequality of arithmetic mean, Bayes estimator, unbiased estimate.

Classification AMS(1991) 62G05/CCL O212

1. Introduction

If Ψ is a positive random variable and the expectation of Ψ exists, then the arithmetic-harmonic inequality implies

$$E\Psi \geq (E\Psi^{1}) \quad \text{or} \quad E\Psi^{1} \geq (E\Psi)^{-1}. \tag{1}$$

We will generalize this inequality to matrix. Now if X is a positive definite random matrix, and EX is finite, then

$$EX \geq (EX^{-1})^{-1} \quad \text{or} \quad EX^{-1} \geq (EX)^{-1}. \tag{2}$$

This is equivalent to saying that $EX - (EX^{-1})^{-1}$ and $EX^{-1} - (EX)^{-1}$ are nonnegative definite matrices.

By the above inequality, the optimal properties of Bayes unbiased estimates may be proved. As in the book [1], the following results are known.

Let θ be a parameter and t be a statistic with

$$E\{t|\theta\} = \theta, \quad E\{\theta|t\} = t \tag{3}$$

and Et^2 is finite. Then

$$\theta = t \quad \text{a.s.}, \tag{4}$$

i.e., the t is an ideal estimate of θ. In the same book, it is pointed out that the finiteness of Et^2 is not necessary. In [2], the following results are proved.

* Received April 6, 1995.

1. If $E|t| < \infty$, then (4) follows from (3).
2. If $t \geq 0$ a. s., then (4) follows from (3).

In this paper, we extend the above results to the case that θ and t are matrices.

2 An extension of the inequality

For a matrix A, if it is non-negative, then it is denoted by $A \geq 0$. So $A \geq B$ means that $A - B \geq 0$.

Lemma 2.1 *If X is a $p \times p$ random positive definite matrix with finite EX and EX^{-1}, then*

$$EX \geq (EX^{-1})^{-1} \quad \text{or} \quad EX^{-1} \geq (EX)^{-1}. \tag{5}$$

Proof Since for any matrix Y, $Y^TXY \geq 0$. Now let $Y = (EX)^{-1} - X^{-1}$, then

$$0 \leq ((EX)^{-1} - X^{-1})^T X ((EX)^{-1} - X^{-1}) = (EX)^{-1} X (EX)^{-1}) + X^{-1} - 2(EX)^{-1}.$$

Take expectation in both sides of the inequality we get

$$0 \leq E(EX)^{-1} X (EX)^{-1}) + EX^{-1} - 2(EX)^{-1} = EX^{-1} - (EX)^{-1},$$

$$0 \leq (EX)^{-1} + EX^{-1} - 2(EX)^{-1}.$$

The proof has been completed.

From the above proof we have the following corollaries.

Corollary 2.1 $(EX)^{-1} = EX^{-1}$ *iff* $X = EX$ *a. s.*

Corollary 2.2 *The determinants of EX and EX^{-1} are related by* $\det(EX^{-1})\det(EX) \geq 1$.

Corollary 2.3 $E\{X^{-1} | \mathbf{F}\} \geq (E\{X | \mathbf{F}\})^{-1}$ *for any σ-field $\mathbf{F}$.*

3 Bayes unbiased estimator

Now we consider the case where θ and $t \in R^p$.

Theorem 3.1 *If Ett^T is finite and*

$$E\{t|\theta\} = A\theta, \quad E\{\theta|t\} = Bt \text{ with } BA = I_p, \tag{6}$$

then

$$\theta = Bt \quad a. s. \tag{7}$$

Proof By direct computations we have

$$B(Ett^T)B = BE(tE\{\theta^T|t\}) = BE(E\{\theta^T|t\}) = BE(t\theta^T) = E(Bt\theta^T).$$

On the other hand,

$$B(Ett^T)B = E(Bt\theta^T)^T = E(\theta t^T B^T) = E(\theta E\{t^T|\theta\}B^T) = E(\theta\theta^T A^T B^T) = E\theta\theta^T.$$

Now combining the above equations we conclude that

$$E(\theta - Bt)(\theta - Bt)^T = E\theta\theta^T - EBt\theta^T - E\theta t^T B^T - EBtt^T B^T = 0. \tag{8}$$

This results in $\theta = Bt$ a. s. The proof has been completed.

In the linear model, $E\{y|\theta\} = A\theta$ and the corresponding least estimator of θ is

$$(A^T A)^{-1} A^T = : By,$$

where B satisfies that $BA = I$. Now if the second moments of the components of y are finite, then from Theorem 3.1 we drive that

$$\theta = By \quad \text{a. s.} \tag{9}$$

Note that we may get a similar conclusion if θ and t in the Theorem 3.1 are matrices. The following lemma follows directly from the arithmetic-harmonic inequality.

Lemma 3.1 *If $A \in R^{p\times p}$ is a positive definite matrix satifying that $\frac{1}{p}\operatorname{tr}(A) = 1$ and $\det(A) = 1$, then $A = I$.*

Theorem 3.2 *If Θ and $T \in R^{p\times p}$ are positive definite matrices satisfying*

$$E\{\Theta|T\} = T \quad \text{and} \quad E\{T|\Theta\} = \Theta, \tag{10}$$

then $\Theta = T$ a. s.,

Proof From (10), we get

$$E(\Theta^{-1}T) = E(E\{\Theta^{-1}T|\Theta\}) = I,$$

and

$$E(T^{-\frac{1}{2}}\Theta T^{-\frac{1}{2}}) = I.$$

Now let $X = T^{-\frac{1}{2}}\Theta T^{-\frac{1}{2}}$. It is clear that $X^{-1} = T^{\frac{1}{2}}\Theta^{-1}T^{\frac{1}{2}}$ and $EX = I$. Hence we have

$$1 = \frac{1}{p}\operatorname{tr}E(\Theta^{-1}T) = \frac{1}{p}\operatorname{tr}E(T^{\frac{1}{2}}\Theta^{-1}T^{\frac{1}{2}}) = \frac{1}{p}\operatorname{tr}EX^{-1}$$

$$\geq [\det(EX^{-1})]^{\frac{1}{p}} \geq [\det(EX)]^{-\frac{1}{p}} \geq [\det(I)]^{-\frac{1}{p}} = 1.$$

So we have proved that $EX^{-1} = I = EX$. By the Corollary 2.1 we obtain $X = I$ a. s.. The proof has been completed.

References

[1] D. Blackwell and M. A. Girshick, *Theory of games and statistical decisions*, John Wiley, 1954.

[2] P. J. Brickel and C. L. Mallows, *A note on unbiased Bayes estimates*, Statistical Rs Reports, AT &T, Bell Laboratory, **49** (1987).