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# Universal Residual Intersection *

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**Abstract** This paper devises the theory of universal residual intersection. The mainresult is that in an extension of $R$ of the form $S = R(X) = R[X]_m R[X]$, there is an $s$ - residual intersection $URI(s;I)$ of $IS$, it has the properties that given any $s$ - residual intersection $J$ of $I$ in $R$, $URI(s;I)$ is essentially a deformation of $J$, and is the localization $RI(s;I)_m R[X]$ of the generic $s$ - residual intersection $RI(s;I)$ of $I$.

**Keywords** generic residual intersection, universal residual intersection.

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## 1. Introduction

In 1972, Artin and Nagata [1] introduced the concept of residual intersection of an ideal, they did not explicitly define this concept but roughly speaking if $X$ is an algebraic variety and $Y$ is a closed subscheme in $X$ contained in a closed scheme $Z$, then a resdual intersection of $Y$ in $Z$ is a closed subscheme $W$ such that $W \cup Y = Z$. Let $X$ and $Y$ be two irreducible closed subschemes of a Noetherian schemes $Z$ with $\mathrm{Codim}_Z X \leq \mathrm{Codim}_Z Y = s$ and $Y \nsubseteq X$, then $Y$ is called a residual intersection of $X$ if the number of equations needed to define $X \cup Y$ as a subscheme of $Z$ is the smallest possible, namely $s$. However, in order to include the case where $X$ and $Y$ are reducible with $X$ possibly containing some component of $Y$, in 1988, Huneke and Ulrich [6] gave the following more general definition.

**Definition 1**[6] *Let $R$ be a Noetherian ring and let $I$ be an ideal of $R$, $s \geq \mathrm{ht}(I)$ and $A = (a_1, \cdots, a_s) \subset I$, with $A \neq I$, set $J = A : I$, if $\mathrm{ht}(J) \geq s$, then $J$ is said to be an $s$ - residual intersection of $I$ (with respect to $A$). If furthermore $I_p = A_p$ for all $p \in V(I) = \{p \in \mathrm{Spec}(R) : I \subset P\}$ with $\mathrm{ht}(p) \leq s$, then we say $J$ is a geometric $s$ - residual intersection of $I$.*

Properties of residual intersection and in particular their Cohen - Macaulayness have been studied in [1], [3], [4], [6]. Further more, Huneke and Ulrich defined the generic residual intersection and studied their properties. Using generic residual intersection, they obtained some beautiful results (See [6], [8]).

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In section two, we devise the theory of universal residual intersection and give some properties and applications. Our main theorem is that in an extension of $R$ of the form $S = R(X) = R[X]_{mR[X]}$, there is a $s$ - residual intersection $URI(s; I)$ of $IS$, it has the properties that given any $s$ - residual intersection $J$ of $I$ in $R$, $URI(s; I)$ is essentially a deformation of $J$, and is the localzition $RI(s; I)_{mR[X]}$ of the generic $s$ - residual intersection $RI(s; I)$ of $I$, thus mativating the study of universal residual intersection.

## 2. Universal residual intersection

In this section, we will briefly develop the theory of universal residual intersection and give some properties and application.

Before we proceed, more definitions are needed.

**Definition 2.1**[4] *An ideal $I$ in a Noetherian local ring $R$ is called Strongly Cohen - Macaulay (abbr. SCM) if all Koszul homology modules of some (and hence every) generating set of $I$ are Cohen - Macaulay (abbr. CM) modules.*

**Examples** (1) In a local CM ring, almost complete intersection CMideals are SCM (see [5] 1.13).

(2) Let $I$ be an ideal of a local Gorenstein ring $R$ and satisfy

(a) $\mu(I) = \mathrm{ht}(I) + 2$,

(b) $I$ is CM,

then Avramov and Herzog[2] proved that $I$ is SCM.

**Definition 2.2**[7] *Let $(R, I)$ and $(S, J)$ be pairs where $R$, $S$ are Noetherian rings, and $I \subset R$, $J \subset S$ are ideals or $I = R$ or $J = S$.*

*(a) $(R, I)$ and $(S, J)$ are isomorphic (Write $\cong$) if there is an isomorphism of rings $\varnothing: R \to S$ with $\varnothing(I) = J$.*

*(b) $(R, I)$ and $(S, J)$ are equivalent (Write $\equiv$) if there are finite sets of variables $X$ and $Z$ such that $(R[X], IR[X]) \cong (S[Z], JS[Z])$.*

*(c)* Let R, S be local, then (R, I) and (S, J) are generically equivalent (Write $\approx$) if there are finite sets of variables X and Z, such that $(R(X), IR(X)) \cong (S(Z), JS(Z))$.

**Definition 2.3**[7] *In addition to the assumption in 2.2, let $R$, $S$ be local,*

*(1) $(S, J)$ is a deformation of $(R, I)$ if there is a sequence $a = a_1, \cdots, a_n$ in $S$ which is regular on $S$ and $S/J$ such that $(S/(a), (J, a)/(a)) \cong (R, I)$.*

(2) *$(S, J)$ is essentially a deformation of $(R, I)$ if there is a sequence of pairs $(S_i, J_i)$, $1 \leq i \leq n$, such that $(S_1, J_1) = (R, I)$, $(S_n, J_n) = (S, J)$ and for any $1 \leq i \leq n$, one of the following conditions is satisfied:*

*(a) $(S_{i+1}, J_{i+1}) = ((S_i)_p, (J_i)_p)$ for some $p \in \mathrm{Spec}(S_i)$,*

*(b) $(S_{i+1}, J_{i+1})$ is a deformation of $(S_i, J_i)$,*

*(c) $(S_{i+1}, J_{i+1}) \approx (S_i, J_i)$.*

The main focus in this section is universal and generic residual intersection, whose definitions we now define and recall.

**Definition 2.4** *Let $R$ be a Noetherian ring, let $I = R$ with $s \geq 1$, or let $I \neq 0$ be an $R$ - ideal satis-*

*fying* $G_{s+1}$ *where* $s \geq \max\{1, \mathrm{ht}(I)\}$, *Further let* $\underline{f} = f_1, \cdots, f_n$ *be a generating set of* $I$ *and let* $X$ *be a generic* $s \times n$ *matrix and set* $(\underline{a})^t = X(f_1, \cdots, f_n)^t$ *where the "* $t$ *" stands for the transpose.*

*(a)* $RI(s;\underline{f}) = (\underline{a})R[X] : IR[X] \subset R[X]$ *is called the generic* $s$ *- residual intersection of* $I$ *with respect to the generating set* $\underline{f}$. *([6] 3.1).*

*(b) Let* $(R, m)$ *be local,* $R(X) = R[X]_{mR[x]}$,

$URI(s;\underline{f}) = (\underline{a})R(X) : IR(X) \subset R(X)$ *is called the universal* $s$ *- residual intersection of* $I$ *with respect to* $\underline{f}$.

**Remark 2.5** Generic and universal residual intersection can be viewed as the natural generalization of generic and universal linkage. From their definitions, we know that if $R$ is a local CM ring, $I$ is an $R$ - ideal, the generic $ht(I)$ - residual intersection coincides with the first generic linkage; the universal $ht(I)$ - residual intersection coincides with the first universal linkage.

**Proposition 2.6** *Let* $R$, $I$, $s$ *be as in definition 2.4, choose two generating sequences* $\underline{f} = f_1, \cdots, f_n$ *and* $\underline{h} = h_1, \cdots, h_m$ *of* $I$, *let* $X$ *be a generic* $s \times n$ *matrix and* $Z$ *be a generic* $s \times m$ *matrix, Then*

*(a)* $(R[X], RI(s;\underline{f})) \equiv (R[Z], RI(s;\underline{h}))$, *Moreover the isomorphism defining this equivalence is* $R$ *- linear (See [8].2.2);*

*(b)* $(R(X), URI(s;\underline{f})) \approx (R(Z), URI(s;\underline{h}))$.

**Proof** We only prove (b). Since $\underline{f}$ and $\underline{h}$ are generators of $I$, we may suppose that $\underline{f} \subset \underline{h}$, and by induction ,it even suffices to show the result for $\underline{h} = \underline{f}, h$, Write

$h = \sum_{j=1}^n r_j f_j$, for $r_j \in R$ and define an $R$ - isomorphism;

$\varnothing : R[X, Z] \to R[X, Z]$ with

$$\varnothing(X_{ij}) = Z_{ij} + r_j Z_{i(n+1)}, \text{ for } 1 \leq i \leq s,\ 1 \leq j \leq n,$$

$$\varnothing(Z_{ij}) = X_{ij}, \text{ for } 1 \leq i \leq s,\ 1 \leq j \leq n,$$

$$\varnothing(Z_{in+1}) = Z_{i(n+1)}, \text{ for } 1 \leq i \leq s.$$

Obviously $\varnothing$ can be extended to an automorphism $\psi$ on $S = R(X, Z)$. Observe that for all $1 \leq i \leq s$,

$$\psi\left(\sum_{j=1}^n X_{ij} f_j\right) = \sum_{j=1}^n (Z_{ij} + r_j Z_{i(n+1)}) f_j = \sum_{j=1}^n Z_{ij} f_j + Z_{i(n+1)} \sum r_j f_j$$

$$= \sum_{j=1}^n Z_{ij} f_j + Z_{i(n+1)} h.$$

Thus if we write

$$(\underline{a})^t = (a_1, \cdots, a_s)^t = X(f_1, \cdots, f_n)^t,$$

$$(\underline{b})^t = (b_1, \cdots, b_s)^t = Z(f_1, \cdots, f_n, h)^t,$$

then $\psi(a_i) = b_i$ for $1 \leq i \leq s$, since $\psi(IS) = IS$, it follows that

$$\psi(URI(s; f_1, \cdots, f_n) S) = \psi((\underline{a}) S : IS)$$

$$= (\psi(\underline{a}) S) : IS = (\underline{b}) S : IS = URI(s; f_1, \cdots, f_n, h) S.$$

Therefore, we have proved that

$$(R(X), URI(s;\underline{f})) \approx (R(Z), URI(s;\underline{h})). \quad \#$$

This proposition shows that up to equivalence of pairs in the sence of 2.2 (b), the definition of $(R[X], RI(s;\underline{f}))$ only depends on $I$ and $s$, but not on the choice of $\underline{f}$. Hence we write $RI(s;I)$ instead of $RI(s;\underline{f})$. We call $RI(s;I)$ the generic $s$-residual intersection of $I$. Likewise up to generic equivalence of pairs in the sence of 2.2 (c), $(R(X), URI(s;\underline{f}))$ only depends on $I$ and $s$, but not onthe choice of $\underline{f}$. Hence we write $URI(s;I)$ for $URI(s;\underline{f})$ and call $URI(s;I)$ the universal $s$-residual intersection of $I$.

Next we list some elementary properties of universal residual intersection.

**Proposition 2.7** *Let $R$ be a local CM ring, and $I$ be a CM $R$-ideal satisfying $G_{s+1}$ where $s \geq grade(I) > 0$, let $RI(s;I) \subset R[X]$ be any generic $s$-residual intersection and let $URI(s;I) \subset R(Z)$ be any universal $s$-residual intersection of $I$. Then*

*(a)* $(R(X), RI(s;I)R(X)) \approx (R(Z), URI(s;I))$,

*(b)* $(R_p[X], RI(s;I)R_p[X]) \equiv (R_p[X], RI(s;I_p))$ *for* $p \in \mathrm{Spec}(R)$,

*(c)* $(R_p(Z), URI(s;I)R_p(Z)) \approx (R_p(Z), URI(s;I_p))$ *for* $p \in \mathrm{Spec}(R)$,

*(d)* $RI(s;I)$ *is a geometric $s$-residual intersection of* $IR[X]$,

*(e)* $URI(s;I)$ *is a $s$-residual intersection of* $IR(Z)$.

**Proof** From the definition, we have the following facts:

$(R(X), RI(s;I)R(X)) = (R[X]_{mR[X]}, RI(s;I)_{mR[X]})$ is a universal $s$-residual intersection of $I$;

$(R_p[X], RI(s;I)R_p[X]) = (R[X] \otimes R_p, RI(s;I) \otimes R_p)$ is a generic $s$-residual intersection of $I_p$;

$(R_p(Z), URI(s;I)R_p(Z)) = (R(Z)_pR(Z), URI(s;I)_pR(Z))$ is a universal $s$-residual intersection of $I_p$.

Then claims (a), (b), (c) follows from the above because generic (anduniversal) residual intersections are unique up to (generic) equivalence.

(d) Since $I$ satisfies $G_{s+1}$, Lemma 3.2 in [6] shows that $RI(s;I)$ is actually a geometric $s$-residual intersection of $IR[X]$.

(e) is trival. #

The following observation will play an important role in our later discussion. It clarifies the relations between generic and arbitrary residual intersection, between universal and arbitrary residual intersection, thus motivating the study of universal residual intersection.

**Theorem 2.8** *Let $R$ be a local Gorenstein ring and $I$ be a SCM ideal satisfying $G_{s+1}$, where $s \geq grade(I) \geq 1$, let $RI(s;I)$ be a generic $s$-residual intersection of $I$ in $R[X]$ and let $URI(s;I)$ be a universal $s$-residual intersection of $I$ in $R(X)$, and $J$ be an arbitrary $s$-residual intersection of $I$ in $R$. Then*

*(a)* *There exists $q \in \mathrm{Spec}(R[X])$ such that $(R[X]_q, RI(s;I)_q)$ is a deformation of $(R, J)$ ([8] 1.5).*

*(b)* $(R(X), URI(s;I))$ *is essentially a deformation of* $(R, J)$.

**Proof** We only to prove (b). Part (b) follows from (a), Since $(R(X), URI(s;I)) \approx (R(X), RI$

$(s; I) R(X))$, by proposition 2.7(a) and since $R(X)$ is a localization of $R[X]_q$ with respect to the prime ideal $mR[X]_q$. #

To provide further motavation we are now going to list some rather natural classes of examples which all turn out to be the localization of generic residual intersections.

**Example 2.9** Let $(R, m)$ be a local CM ring, let $I = (f_1, \cdots, f_n)$ be a SCM ideal of grade $g \geq 1$ satisfying $G_\infty$, then in $T = R[X_1, \cdots, X_n, V]$,

$$RI(n;(I,V)) = (\sum_{j=1}^{n} b_i X_i : \sum_{j=1}^{n} b_i f_i = 0, VX_i - f_i : i = 1, \cdots, n)$$

and $T/RI(n;(I,V)) \cong R[It, t^{-1}]$ ( See [4] ). therefore

$$URI(n;(I,V)) = RI(n;(I,V))_{mR[X_1, \cdots, X_n, V]}$$

$$= (\sum_{j=1}^{n} b_i X_i : \sum_{j=1}^{n} b_i f_i = 0, VX_i - f_i : i = 1, \cdots, n)_{mR[X_1, \cdots, X_n, V]}.$$

**Example 2.10** Let $(R, m)$ be a local CM ring, let $Z$ be a generic $(r-1) \times r$ matrix ( $r \geq 2$) with maximal minors $\triangle_1, \cdots, \triangle_r$, let $Y$ be a generic $(s - r + 1) \times r$ matrix ( $s > r$), and let $X$ be the generic $s \times r$ matrix $\left(\frac{Z}{Y}\right)$, Huneke and Ulrich [8] studied the generic $(s - r + 1)$ - resdual intersection $RI(s - r + 1; \triangle_1, \cdots, \triangle_r) \subset R[X]$ of the generic perfect grade 2 ideal $I_{r-1}(Z) \subset R[Z]$,

$$RI(s - r + 1; I_{r-1}(Z)) = RI(s - r + 1; (\triangle_1, \cdots, \triangle_r)) = I_r(X),$$

therefore $URI(s - r + 1; I_{r-1}(Z)) = I_r(X)_{mR[X]}$.

**Example 2.11** Let $(R, m)$ be a local CM ring, let $y_1, \cdots, y_g$ be a regular sequence in $R$, let $X$ be a generic $s \times g$ matrix ( $s > g$), let

$$(a_1, \cdots, a_s)^t = Z(y_1, \cdots, y_g)^t.$$

Then in $R[X]$,

$$RI(s; y_1, \cdots, y_g) = (a_1, \cdots, a_s, I_g(X)) \quad \text{(See [6])},$$

$$URI(s; y_1, \cdots, y_g) = (a_1, \cdots, a_s, I_g(X)) R(X) = (a_1, \cdots, a_s, I_g(X))_{mR[X]}.$$

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# 泛剩余交

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## 摘　要

**本文建立了泛剩余交理论,并揭示了它与剩余交和一般剩余交的关系,得到:在交换局部环 $R$ 的扩张 $S = R(X) = R[X]_{mR[X]}$ 中,存在 $IS$ 的一个 $s$ - 剩余交 $URI(s;I)$,使得对 $I$ 在 $R$ 中的任意 $s$ - 剩余交 $J$, $URI(s;I)$ 是 $J$ 的本质形变,且是 $I$ 的一般 $s$ - 剩余交 $RI(s;I)$ 局部化 $RI(s;I)_{mR[X]}$. 并给出了一些应用,为研究剩余交和一般剩余交提供了工具.**