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On Absolute Convergence of Bernstein Polynomials *

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Abstract This note is devoted to the study of the absolute convergence of Bernstein polynomials. It is proved that for each $x \in [0,1]$, the sequence of the Bernstein polynomials of a function of bounded variation is absolutely summable by $|C,1|$ method. Moreover, the estimate of the remainders of the $|C,1|$ sum of the sequence of the Bernstein polynomials is obtained.

Keywords Bernstein polynomial, absolute convergence.

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The Bernstein polynomials $B_n(f;x)$ of a function f defined on $[0,1]$ are

$$B_n(f;x) \overset{\text{def}}{=} \sum_{j=0}^{n} f\left(\frac{j}{n}\right) p_{n,j}(x), \quad p_{n,j}(x) = \binom{n}{j} x^j (1-x)^{n-j}.$$

A classical result is that if f is of bounded variation on $[0,1]$, then for each x in $(0,1)$, $\lim_{n\to\infty} B_n(f;x) = \frac{1}{2}(f(x+0) + f(x-0))$. A quantitative eastimate was proved by Cheng[1] and refined by Chen and Guo[2]. Meanwhile, it is well known that the sequence $\{B_n(f;x)\}$ of Bernstein polynomials is monotonic with respect to n whenever f is a convex function, that means $B_n(f;x) \le B_{n-1}(f;x)$, so that $\{B_n(f;x)\}$ is absolutely convergent, namely

$$\sum_{n=2}^{\infty} |B_{n-1}(f;x) - B_n(f;x)| < \infty \tag{1}$$

holds at x where $B_n(f;x)$ converges. In addition, we have the following simple conclusion.

Proposition *If $f \in C^1[0,1]$ and $\sum n^{-1}\omega(f', n^{-1}) < \infty$, then $\{B_n(f;x)\}$ is absolutely convergent for each $x \in [0,1]$.*

Proof An elementary computation shows that

$$B_{n-1}(f;x) - B_n(f;x) = \sum_{j=1}^{n-1}\left[\frac{j}{n} f\left(\frac{j-1}{n-1}\right) - f\left(\frac{j}{n}\right) + \left(1-\frac{j}{n}\right) f\left(\frac{j}{n-1}\right)\right] p_{n,j}(x)$$

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$$= \sum_{j=1}^{n-1}\Big[\frac{j}{n}\int_{\frac{j-1}{n-1}}^{\frac{j}{n}}\Big(f'\Big(\frac{j}{n}\Big) - f'(t)\Big)\,dt + \Big(1 - \frac{j}{n}\Big)\int_{\frac{j}{n}}^{\frac{j}{n-1}}\Big(f'(t) - f'\Big(\frac{j}{n}\Big)\Big)\,dt\Big]p_{n,j}(x)$$

$$\le \sum_{j=1}^{n-1}\frac{j(n-j)}{n^2(n-1)}\Big[\omega\Big(f';\frac{n-j}{n(n-1)}\Big) + \omega\Big(f';\frac{j}{n(n-1)}\Big)\Big]p_{n,j}(x)$$

$$\le 2n^{-1}\omega(f';n^{-1})\sum_{j=1}^{n-1}p_{n,j}(x) \le 2n^{-1}\omega(f';n^{-1})$$

which proves (1) under the condition that $\sum n^{-1}\omega(f';n^{-1}) \le \infty$. □

It is impossible to convince that (1) holds for general functions, even for that of bounded variation. However, it is meaningful to consider the absolute summability of Bernstein polynomials in some weaker sense, for example, the $|C,1|$ summability. In fact, we shall give an estimate for the remainders of the $|C,1|$ sums of the Bernstein polynomials of functions of bounded variation.

Theorem *Let f be of bounded variation on $[0,1]$. Then for each $x \in [0,1]$, $\{B_n(f;x)\}$ is absolutely summable by the Cesàro method $|C,1|$ of order 1. Moreover, for $x \in (0,1)$*

$$R_n(f,x) \le M(x)\Big\{\frac{1}{n}\sum_{k=1}^{n}V(g_x)\Big|_{x-x/\sqrt{k}}^{x+(1-x)/\sqrt{k}}\log\Big(2+\frac{n}{k}\Big) + \frac{|f(x+0)-f(x)|+|f(x-0)-f(x)|}{\sqrt{n}}\Big\}, \tag{2}$$

where

$$R_n(f;x) = \sum_{m=n}^{\infty}|\sigma_{m-1}(f;x) - \sigma_m(f;x)|,$$

$$\sigma_m(f;x) = \frac{1}{m}\sum_{k=1}^{m}B_k(f;x),$$

$$g_x(t) = \begin{cases} f(t) - f(x+0), & t > x; \\ 0, & t = x; \\ f(t) - f(x-0), & t < x, \end{cases}$$

Set

$$K_n(x,t) = \begin{cases} \sum_{v \le tn} p_{n,v}(x), & 0 < t \le 1; \\ 0, & t = 0, \end{cases}$$

Proof of the Theorem If $x=0$ or 1, (2) is obvious since $B_n(f;0)=f(0)$ and $B_n(f;1)=f(1)$. For a fixed $x\in(0,1)$, writting

$$f(t)-f(x)=g_x(t)+(f(x+0)-f(x))(t-x)_+^0+(f(x-0)-f(x))(x-t)_+^0,$$

where $(u)_+^0=1$ if $u>0$ and $(u)_+^0=0$ if $u\le 0$, we have

$$R_n(f;x)\le R_n(g_x,x)+|f(x+0)-f(x)|\,R_n((\cdot-x)_+^0;x)$$
$$+|f(x-0)-f(x)|\,R_n((x-\cdot)_+^0;x).$$

It follows from (3) and (4) that

$$\begin{aligned}R_n((x-\cdot)_+^0;x)&=\sum_{m=n}^{\infty}\frac{1}{m(m-1)}\Big|\,mB_m((x-\cdot)_+^0;x)-\sum_{\nu=1}^{m}B_\nu((x-\cdot)_+^0;x)\Big|\\&=\sum_{m=n}^{\infty}\frac{1}{m(m-1)}\Big|\,m\sum_{k<mx}p_{m,k}(x)-\sum_{\nu=1}^{m}\sum_{k<\nu x}p_{\nu,k}(x)\Big|\\&=\sum_{m=n}^{\infty}\frac{1}{m(m-1)}\Big|\,m\Big(\frac12+O_x(m^{-1/2})\Big)-\sum_{\nu=1}^{m}\Big(\frac12+O_x(\nu^{-1/2})\Big)\Big|\\&=O_x\Big(\sum_{m=n}^{\infty}m^{-3/2}\Big)=O_x(n^{-1/2}),\end{aligned}$$

and by the same way, $R_n((\cdot-x)_+^0;x)=O_x(n^{-1/2})$.

Writting

$$R_n(g_x;x)=\int_0^1 g_x(t)\,d_tK_n(x,t),\tag{5}$$

if follows that

$$\begin{aligned}R_n(g_x;x)&=\sum_{m=n}^{\infty}\frac{1}{m(m-1)}\Big|\,mB_m(g_x;x)-\sum_{\nu=1}^{m}B_\nu(g_x;x)\Big|\\&=\sum_{m=n}^{\infty}\frac{1}{m(m-1)}\Big|\int_0^1 g_x(t)\,d_tA_m(x,t)\Big|.\end{aligned}$$

With a special understand to the integration of the Lebesgue - Stieltjes "type" and after integration by parts, we find that

$$R_n(g_x;x)=\sum_{m=n}^{\infty}\frac{1}{m(m-1)}\Big|\int_0^1 A_m(x,t)\,dg_x(t)\Big|\le\int_0^1\sum_{m=n}^{\infty}\frac{|A_m(x,t)|}{m(m-1)}\,|dg_x(t)|.\tag{6}$$

Making use of the lemma, we have

$$\begin{aligned}\sum_{m=n}^{\infty}\frac{|A_m(x,t)|}{m(m-1)}&\le 2\sum_{m\le|x-t|^{-2}}m^{-2}[\,|A_m(x,x)|+|A_m(x,t)-A_m(x,x)|\,]\\&\quad+2\sum_{m>|x-t|^{-2}}m^{-2}|A_m(x,t)|\\&=O_x\Big\{\sum_{m\le|x-t|^{-2}}m^{-2}(m^{1/2}+m^{3/2}|x-t|)+\sum_{m>|x-t|^{-2}}\frac{\log(m|x-t|^2+2)}{m^2|x-t|^2}\Big\}\\&=O_x(1),\quad \text{uniformly for } t\in[0,1],\ t\neq x,\end{aligned}$$

$$\sum_{m=n}^{\infty}\frac{|A_m(x,x)|}{m(m-1)} = O_x\{\sum_{m=n}^{\infty}m^{-3/2}\} = O_x(n^{-1/2})$$

and

$$\sum_{m=n}^{\infty}\frac{|A_m(x,t)|}{m(m-1)} = O_x\{\sum_{m=n}^{\infty}\frac{\log(m|x-t|^2+2)}{m^2|x-t|^2}$$

$$= O_x\{\frac{\log(n|x-t|^2+2)}{n|x-t|^2}, \quad \text{uniformly for } t\in[0,1], t\neq x.$$

Substituting these into (6) and integrating by parts, it follows that

$$R_n(g_x;x) = O_x\{\int_0^{x-\frac{x}{\sqrt{n}}}\frac{\log(n(x-t)^2+2)}{n(x-t)^2}d(-V(g_x)\Big|_t^x) + \int_{x-\frac{x}{\sqrt{n}}}^{x+\frac{1-x}{\sqrt{n}}}|dg_x(t)|$$

$$+ \int_{x+\frac{1-x}{\sqrt{n}}}^{1}\frac{\log(n(t-x)^2+2)}{n(t-x)^2}d(V(g_x)\Big|_x^t)\}$$

$$= O_x\{\frac{1}{n}V(g_x)\Big|_0^x + \{\int_0^{x-\frac{x}{\sqrt{n}}}\frac{\log(n(x-t)^2+2)}{n(t-x)^2}V(g_x)\Big|_t^x dt + V(g_x)\Big|_{x-\frac{x}{\sqrt{n}}}^{x+\frac{1-x}{\sqrt{n}}}$$

$$+ \frac{1}{n}V(g_x)\Big|_x^1 + \int_{x+\frac{1-x}{\sqrt{n}}}^{1}\frac{\log(n(t-x)^2+2)}{n(t-x)^2}V(g_x)\Big|_x^t dt\}.$$

Taking the substitutions $t = x - \frac{x}{\sqrt{n}}$ and $t = x + \frac{1-x}{\sqrt{n}}$ in the integrals $\int_0^{x-x/\sqrt{n}}$ and $\int_{x+(1-x)/\sqrt{n}}^1$ respectively, we get

$$R_n(g_x;x) = O_x\{\frac{1}{n}V(g_x)\Big|_0^1 + V(g_x)\Big|_{x-\frac{x}{\sqrt{u}}}^{x+\frac{1-x}{\sqrt{u}}} + \frac{1}{n}\int_1^n V(g_x)\Big|_{x-\frac{x}{\sqrt{u}}}^{x+\frac{1-x}{\sqrt{u}}}\log(2+\frac{n}{u})\,du\}$$

$$= O_x\{\frac{1}{n}\sum_{k=1}^{n}V(g_x)\Big|_{x-\frac{x}{\sqrt{k}}}^{x+\frac{1-x}{\sqrt{k}}}\log(2+\frac{n}{k})\}.$$

This completes the proof of the theorem. □

Now we give an explanation to the Lebesgue - Stieltjes integral in (5), which makes the computations in proving the theorem reasonable. For $u\in(0,1)$,

$$\int_{u-}^{u} g_x(t)\,d_tK_n(x,t) = \int_{u-}^{u+} g_x(t)\,d_tK_n(x,t) = g_x(u)(K_n(x,u) - K_n(x,u-)),$$

$$\int_{1-}^{1} g_x(t)\,d_tK_n(x,t) = g_x(1)p_{n,n}(x), \int_0^{0+} g_x(t)\,d_tK_n(x,t) = g_x(0)p_{n,0}(x),$$

$$\int_{u-}^{u} K_n(x,t)\,dg_x(t) = K_n(x,u-)(g_x(u) - g_x(u-)),$$

$$\int_0^{0+} K_n(x,t)\,dg_x(t) = p_{n,0}(x)(g_x(0+) - g_x(0)).$$

Remark It is interesting to compare the nature of the absolute summability of Bernstein polynomials with that of Fourier series. The classical results on the absolute summability of Fourier series are the following three conclusions (cf. [4, pp. 240 - 242], [5] and [6, p. 273]):

(i) (Bernstein) If $f \in \mathrm{Lip}\,\alpha$ for $\alpha > 1/2$, then its Fourier series is absolutely convergent;

(ii) (Zygmund) If f is of bounded varation and $f \in \mathrm{Lip}\,\alpha$ for $\alpha > 0$, then its Fourier series is absolutely convergent;

(iii) (Bosanquet) If f is of bounded variation, then its Fourier series is absolutely summable by Cesáro methods of any positive order.

The conclusion of the theorem of this paper is only partly the analogue of Bosanquet's result. It is interesting to consider the complete analogues of the previous (i), (ii) and (iii) to Bernstein polynomials. It is conjectured that the best conclusion corresponding to Bosanquet's result (iii) would be that "if f is of bounded variation on $[0,1]$, then the sequence of its Bernstein polynomials is absolutely summable by Cesáro methods $|C, \alpha|$ for any $\alpha > 1/2$".

References

[1] Cheng Fuhua, *On the rate of convergence of Bernstein polynomials of functions of bounded variation*, J. Approx. Theory, **39** (1983), 259 - 274.

[2] Chen Wenzhong and Guo Shunsheng, *On the rate of convergence of certain linear positive operators for functions of bounded variation (in Chinese)*, Numerical Mathematics, A Journal of Chinese Universities, **9** (1987), 243 - 252.

[3] G. G. Lorentz, *Bernstein Polynomials*, Mathematical Expositions, Vol. 8, 1953.

[4] A. Zygmund, *Trigonometric Series*, Vol. I, 2nd edition, Cambridge Univ. Press, New York, 1959.

[5] L. S. Bosanquet, *Note on the absolute summability (C) of a Fourier seris*, J. London Math. Soc., **11** (1936), 11 - 15.

[6] Chen Jiangong, *Theory of Trigonometric Series (in Chinese)*, Vol. I, Shanghai Sci. and Tech. Press, Shanghai, 1964.

关于 Bernstein 多项式的绝对收敛性

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摘 要

该文研究 Bernstein **多项式的绝对收敛性. 证明了, 对每个** $x \in [0,1]$, **一个有界变差函数的** Bernstein **多项式序列是绝对** $|C,1|$ **可和的, 而且给出了** Berstein **多项式序列的绝对** $|C,1|$ **和式的余项的估计**.