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On Properly Divergent Series *

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Abstract In this paper, the classical concept of properly divergent is generalized, thereby a theorem of power series is extended into a very general case, and its applications in various series of complex functions are discussed.

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In [1, p215] there is a known result as follows[1]

Let a power series of real variable $\sum_0^\infty a_n x^n$ have a radius of convergence equal to 1. If a_n is real for all values of n and $\sum_0^\infty a_n$ is properly divergent, i.e. $S_n = \sum_0^n a_k \to +\infty$, then for its sum function $f(x)$, $\lim_{x\to 1-0} f(x) = +\infty$.

We shall give a generalization of this theorem.

Let D_ε denote the domain $-\frac{\pi}{2}+\varepsilon < \arg z < \frac{\pi}{2}-\varepsilon\ (\varepsilon \geq 0)$. The bounds of the terms "O" are the absolute constants, the bounds of the terms "O_z" are the constants which are only dependent of z.

Definition *Let $\sum_0^\infty t_n$ be a complex series and $\{T_n\}$ its partial sums. If the sequence $\{T_n\}$ satisfies*

$$T_n \to \infty; \quad T_n = O(n^{2-\delta}) \quad (0 < \delta < 1);$$

$$T_n \in D_\varepsilon \quad (n \geq \overline{N}),$$

then the series $\sum_0^\infty t_n$ is said to be properly divergent with a parameter ε.

Based on this definition, we give the following theorem.

Theorem 1 Let a series of complex functions $f(x) = \sum_0^\infty \omega_n(z)$ be convergent in a damain G and a point set $S \subset G$. Again let a point z_0 on boundary ∂G be a limit point of the set S. Suppose that the functions $\omega_n(z)$ satisfy the following three conditions:

$$\lim_{z\to z_0,\, z\in S} \omega_n(z) = \omega_n(z_0),$$

$$\frac{\omega_n(z)}{\omega_n(z_0)} = O_z\left(\frac{1}{n^3}\right), \quad z \in S,$$

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$$\triangle\left(\frac{\omega_n(z)}{\omega_n(z_0)}\right) := \frac{\omega_n(z)}{\omega_n(z_0)} - \frac{\omega_{n+1}(z)}{\omega_{n+1}(z_0)} = P_n(z)Q_n(z) + O\left(\frac{1}{n^3}\right), \quad z \in S, \tag{5}$$

with the double limit

$$\lim_{\substack{n\to\infty \\ z\to z_0,\, z\in S}} Q_n(z) = 1 \text{ and } P_n(z) > 0, \ z \in S, \tag{6}$$

then if at this point z_0, the series $\sum_0^\infty \omega_n(z)$ is properly divergent with a parameter $\varepsilon > 0$, we have $\lim_{z\to z_0,\, z\in S} f(z) = \infty$.

Proof Write

$$\sum_0^n \omega_k(z_0) = S_n(z_0) = \rho_n e^{i\phi_n}, \quad S_{-1}(z_0) = 0. \tag{7}$$

Because the series $\sum_0^\infty \omega_n(z_0)$ is properly divergent with a parameter $\varepsilon > 0$, from (1), (2) we know for a given $G > 0$, there exists N_1 such that when $n \geq N_1 > \overline{N}$,

$$\rho_n > G, \rho_n = O(n^{2-\delta})\ (0 < \delta < 1), \quad |\phi_n| \leq \frac{\pi}{2} - \varepsilon\ (\varepsilon > 0).$$

Combining (7), (8) with (4), we see easily that

$$S_n(z_0)\frac{\omega_n(z)}{\omega_n(z_0)} = O_z\left(\frac{1}{n^{1+\delta}}\right) \quad (n \geq N_1, \ z \in S),$$

so the series $\sum_0^\infty S_n(z_0)\frac{\omega_n(z)}{\omega_n(z_0)}$ is convergent on S. Using Abel transformation, we get

$$f(z) = \sum_0^\infty (S_n(z_0) - S_{n-1}(z_0))\frac{\omega_n(z)}{\omega_n(z_0)} \tag{9}$$

$$= \sum_0^\infty S_n(z_0)\left(\frac{\omega_n(z)}{\omega_n(z_0)} - \frac{\omega_{n+1}(z)}{\omega_{n+1}(z_0)}\right)$$

$$= \sum_N^\infty S_n(z_0)\triangle\left(\frac{\omega_n(z)}{\omega_n(z_0)}\right) + \sum_0^{N-1} S_n(z_0)\triangle\left(\frac{\omega_n(z)}{\omega_n(z_0)}\right), \quad z \in S, \tag{10}$$

here N is a natural number to be determined later.

Let $f_N(z) = \sum_N^\infty S_n(z_0)\triangle\left(\frac{\omega_n(z)}{\omega_n(z_0)}\right)$. From (5), (7) and (8) it follows that

$$|f_N(z)| = \left|\sum_N^\infty S_n(z_0)P_n(z)Q_n(z)\right| + O(1), \quad z \in S. \tag{11}$$

However by the known condition (6), we can get $\lim_{\substack{n\to\infty \\ z\to z_0,\, z\in S}} \mathrm{Re}(Q_n(z)) = 1$ and $\lim_{\substack{n\to\infty \\ z\to z_0,\, z\in S}} \mathrm{Im}(Q_n(z)) = 0$, so there exist $N_2 \geq N_1$ and η such that when $n \geq N_2$, $|z - z_0| < \eta\ (z \in S)$,

$$\mathrm{Re}(Q_n(z)) > 0; \quad |\mathrm{Im}(Q_n(z))| \leq \frac{\sin\varepsilon}{2}\mathrm{Re}(Q_n(z)),$$

here ε is stated as above. From this and (7) ,(8). It follows that

$$\begin{aligned}\mathrm{Re}\{S_n(z_0)Q_n(z)\} &\geq \rho_n\{\cos\phi_n\} Re(Q_n(z)) - |\mathrm{Im}(Q_n(z))|\} \\ &\geq \rho_n\{\sin\varepsilon Re(Q_n(z)) - \frac{\sin\varepsilon}{2}\} Re(Q_n(z))\} \\ &\geq \frac{\sin\varepsilon}{2} GRe(Q_n(z)) > 0 \ (n \geq N_2, |z - z_0| < \eta, z \in S).\end{aligned}$$

Combining this with (6) , we get from (10) when $N \geq N_2$

$$\begin{aligned}|f_N(z)| &\geq \sum_N^\infty P_n(z)\}\mathrm{Re}\{S_n(z_0)Q_n(z)\} + O(1) \\ &\geq \frac{\sin\varepsilon}{2}\mathrm{GRe}\{\sum_N^\infty P_n(z)Q_n(z)\} + O(1) \quad (|z - z_0| < \eta, z \in S).\end{aligned}$$

From this and (5) ,

$$|f_N(z)| \geq \frac{\sin\varepsilon}{2} G\{\mathrm{Re}(\sum_N^\infty \triangle(\frac{\omega_n(z)}{\omega_n(z_0)})) + O(\frac{1}{N^2})\} + O(1)\} \ (N \geq N_2, |z - z_0| < \eta, z \in S). \tag{12}$$

But by (4) - (5) ,

$$\begin{aligned}\sum_N^\infty \triangle(\frac{\omega_n(z)}{\omega_n(z_0)}) &= \frac{\omega_N(z)}{\omega_N(z_0)} - \lim_{n\to\infty}\frac{\omega_n(z)}{\omega_n(z_0)} \\ &= \frac{\omega_N(z)}{\omega_N(z_0)} \quad (z \in S).\end{aligned}$$

Again by (3) ,

$$\lim_{z\to z_0, z\in S}\sum_N^\infty \triangle(\frac{\omega_n(z)}{\omega_n(z_0)}) = 1.$$

Furthermore, by (11) we have

$$\liminf_{z\to z_0, z\in S}|f_N(z)| \geq \frac{\sin\varepsilon}{2} G\{1 + O(\frac{1}{N^2})\} + O(1).$$

If we choose a fixed $N(\geq N_2)$ so large that the expression in the curly brackets $> \frac{1}{2}$, we get finally

$$\liminf_{z\to z_0, z\in S}|f_N(z)| > \frac{\sin\varepsilon}{4} G + O(1) \quad (G \text{ is an arbitrarily large number}).$$

This implies that

$$\lim_{z\to z_0, z\in S} f_N(z) = \infty. \tag{13}$$

On the other hand, for this fixed N , by (3) , we also have

$$\sum_0^{N-1} S_n(z_0)\triangle(\frac{\omega_n(z)}{\omega_n(z_0)}) = o(1) \quad (z \to z_0, z \in S).$$

From this and (12), (9), we obtain that $\lim_{z \to z_0, z \in S} f(z) = \infty$. Theorem 1 is proved.

Next, we shall give the applications of Theorem 1 in various series below.

Let a power series of complex variable $f(z) = \sum_0^\infty a_n z^n$ have a radius of convergence equal to 1 and let $z_0 = e^{i\theta_0}$, $S = :\{z; z = re^{i\theta_0}, \frac{1}{2} \le r < 1\}$. We easily prove that if at z_0 series $\sum_0^\infty a_n z^n$ is properly divergent with a parameter $\varepsilon > 0$, then $\lim_{z \to z_0, z \in S} f(z) = \infty$.

Consider a counterexample: the point $z_0 = e^{i\theta_0}$ and the series

$$i \sum_0^\infty (-1)^n (n+1) \left(\frac{z}{z_0}\right)^n = i\left(1 + \frac{z}{z_0}\right)^{-2} \quad (|z| < 1),$$

we can see the condition $\varepsilon > 0$ can't be improved into $\varepsilon = 0$.

For a trigonometric series of complex variable:

$$\sum_0^\infty (c_n \cos nz + d_n \sin nz) = \sum_{-\infty}^\infty f_n e^{inz}$$

$$\left(f_0 = c_0, f_n = \frac{1}{2}(c_n - i d_n), f_{-n} = \frac{1}{2}(c_n + i d_n), n = 1, 2, \cdots\right),$$

if $\limsup_{n \to \infty} \sqrt[n]{|f_n|}\ n = \lambda (\lambda > 0)$, it is convergent on the half plane $\mathrm{Im} z > \ln \lambda$.

Now take $z_0 = x_0 + i \ln \lambda$, $S = :\{z, z = x_0 + iy, y > \ln \lambda\}$ and write $\omega_n(z) = f_n e^{inz}$. Since $\lim_{x \to z_0, z \in S} \omega_n(z) = \omega_n(z_0)$,

$$\frac{\omega_n(z)}{\omega_n(z_0)} = e^{-n(y - \ln\lambda)} = O_z\left(\frac{1}{n^3}\right), \quad z \in S,$$

$$\triangle\left(\frac{\omega_n(z)}{\omega_n(z_0)}\right) = e^{-n(y-\ln\lambda)}\left(1 - e^{-(y-\ln\lambda)}\right) > 0, \quad z \in S$$

(correspondingly $P_n(z) = e^{-n(y-\ln\lambda)}(1 - e^{-(y-\ln\lambda)})$, $Q_n(z) = 1$), we see the conditions of Theorem 1 are satisfied. Hence if at the point z_0 the series $\sum_0^\infty (c_n \cos nz + d_n \sin nz)$ is properly divergent with a parameter $\varepsilon > 0$, then for its sum function $f(z)$, we have $\lim_{z \to z_0, z \in S} f(z) = \infty$.

For a Bessel series

$$\sum_0^\infty a_n J_n(z), \tag{14}$$

where Bessel function $J_n(z) = \frac{1}{\pi} \int_0^\pi \cos(n\theta - z\sin\theta)\, d\theta$, $z \in C$, $n = 0, 1, 2, \cdots$. If

$$\overline{\lim_{n \to \infty}}\left(\frac{e}{2} \sqrt[n]{|a_n|}\right) = R^{-1}, \tag{15}$$

it can be proved that the series (13) has a circle of convergence $|z| = R$.

Theorem 2 Suppose that $|z| = 1$ is the circle of convergence of Bessel series (13) and $f(z) = \sum_0^\infty a_n J_n(z)$, $|z| < 1$. If at a point $z_0 = e^{i\theta_0}$ the series (13) is properly divergent with a parameter $\varepsilon > 0$, then we have

$$\lim_{r\to 1-0} f(re^{i\theta_0}) = \infty$$

and $\varepsilon > 0$ can't be improved into $\varepsilon = 0$.

Proof Take $G =: \{z; |z| < 1\}$, $S =: \{z; z = re^{i\theta_0}, \frac{1}{2} \le r < 1\}$ and write $\omega_n(z) = a_n J_n(z)$, it is clear that the condition (3) is valid. Using the known formula of Bessel functions[2],[3]

$$J_n(z) = \frac{1}{n!}\left(\frac{z}{2}\right)^n \left\{1 + \sum_{k=1}^{\infty} \frac{(-1)^k n!}{k!(n+k)!}\left(\frac{z}{2}\right)^{2k}\right\} \tag{16}$$

and noticing that

$$\left|\sum_{k=1}^{\infty} \frac{(-1)^k n!}{k!(n+k)!}\left(\frac{z}{2}\right)^{2k}\right| \le \frac{1}{4(n+1)} \sum_{k=1}^{\infty} \frac{1}{k!} \le \frac{3}{4} \quad \left(z = re^{i\theta}, \frac{1}{2} \le r < 1, \ n = 0, 1, \cdots\right),$$

we obtain

$$\frac{r^n}{n!2^{n+2}} \le |J_n(re^{i\theta})| \le \frac{r^n}{n!2^{n-1}} \quad \left(\frac{1}{2} \le r \le 1, \ 0 \le \theta \le 2\pi\right),$$

so $\left|\frac{\omega_n(z)}{\omega_n(z_0)}\right| = \left|\frac{J_n(z)}{J_n(z_0)}\right| \le 8r^n (z \in S)$, i.e., (4) is valid.

By (15), we have

$$\frac{J_n(z)}{J_n(z_0)} = r^n \left\{1 + \frac{e^{2i\theta_0}}{4(n+1)}(1 - r^2) + \frac{e^{4i\theta_0}}{32(n+1)(n+2)}(1 - r^2)^2 + O\left(\frac{1}{n^3}\right)\right\} \quad (z_0 = e^{i\theta_0}, \ z \in S).$$

A direct calculation gives

$$\triangle\left(\frac{\omega_n(z)}{\omega_n(z_0)}\right) = \frac{J_n(z)}{J_n(z_0)} - \frac{J_{n+1}(z)}{J_{n+1}(z_0)} = P_n(z) Q_n(z) + O\left(\frac{1}{n^3}\right) \quad (z_0 = e^{i\theta_0}, z \in S),$$

where

$$P_n(z) = r^n(1 - r), \quad z \in S,$$

$$Q_n(z) = 1 + \frac{e^{2i\theta_0}}{4(n+1)}(1 - r^2) + \frac{e^{4i\theta_0}}{32(n+1)(n+2)}(1 - r^2)^2 + \frac{r(1+r)e^{2i\theta_0}}{4(n+1)(n+2)}, \quad z \in S,$$

it is clear that $\lim_{\substack{n\to\infty \\ z\to z_0, z\in S}} Q_n(z) = 1$ and $P_n(z) > 0, z \in S$, so the condition (5) is valid. Finally, by Theorem 1, $\lim_{r\to 1-0} f(re^{i\theta_0}) = \infty$.

Consider a Bessel series:

$$iO_0'(1) J_0(z) + 2i \sum_{k=1}^{\infty} O_k'(1) J_k(z), \tag{17}$$

where $O_k(z)$ are the Neumann polynomials[2],[3]

$$O_0(z) = \frac{1}{z}, \quad O_k(z) = \sum_{m=0}^{[\frac{k}{2}]} \frac{2^{k-2m-1} k(k-m-1)!}{m! z^{k-2m+1}} \quad (k \ge 1).$$

By the above formula we can get

$$- O'_k(1) = 2^{k-1}(k+1)!\{1 + \frac{1}{4(k+1)} + O(\frac{1}{k^2})\} \quad (k \geq 2). \tag{18}$$

Hereafter the terms $O(\frac{1}{k^2})$ are real value. So by (14) the circle of convergence of the series (16) is $|z| = 1$.

By $S_n(z)$ denote the partial sums of the series (16), so

$$S_n(-1) = iO'_0(1)J_0(-1) + 2i\sum_{k=1}^{n} O'_k(1)J_k(-1). \tag{19}$$

Again from (15) we have

$$J_k(-1) = \frac{(-1)^k}{k!2^k}\{1 - \frac{1}{4(k+1)} + O(\frac{1}{k^2})\} \quad (k \geq 1), \tag{20}$$

Now combining (17) - (19), we have

$$iS_{2m}(-1) = -m - 2 + \sum_{k=2}^{2m} O(\frac{1}{k}) - O'_0(1)J_0(-1) - 2O'_1(1)J_1(-1)$$

and

$$iS_{2m+1}(-1) = m + \sum_{k=2}^{2m+1} O(\frac{1}{k}) - O'_0(1)J_0(-1) - 2O'_1(1)J_1(-1),$$

so there exists N such that $S_n(-1) \in D_0(n \geq N)$ and $S_n(-1) = O(n)$. By the definition, at the point $z_0 = -1$ the series (16) is properly divergent with a parameter $\varepsilon = 0$.

On the other hand, applying a known formula[2],[3]:

$$\frac{1}{1-z} = O_0(1)J_0(z) + 2\sum_{k=1}^{\infty} O_k(1)J_k(z) \quad (|z| < 1)$$

and the derivative formula of Bessel functions[2],[3]

$$J'_0(z) = -J_1(z), \; 2J'_k(z) = J_{k-1}(z) - J_{k+1}(z),$$

we can obtain that

$$\begin{aligned}\frac{1}{(1-z)^2} &= -O_0(1)J_1(z) + \sum_{k=1}^{\infty} O_k(1)(J_{k-1}(z) - J_{k+1}(z)) \\ &= O_1(1)J_0(z) + \sum_{k=1}^{\infty}(O_{k+1}(1) - O_{k-1}(1))J_k(z).\end{aligned}$$

From this and $O'_0(z) = -O_1(z)$, $2O'_k(z) = O_{k-1}(z) - O_{k+1}(z)$[2],[3], we know that the sum function of the series (16) is $-i(1-z)^{-2}$. Hence the point $z_0 = -1$ is its regular point.

To sum up, we know that the condition $\varepsilon > 0$ can't be improved into $\varepsilon = 0$. Theroem 2 is proved.

In this paper, if the above domain D_ε is replaced by the domain:

$$\beta_1 < \arg(z - a) < \beta_2,\ 0 < \beta_2 - \beta_1 < \pi - \varepsilon,$$

where a is complex number and β_1, β_2 are real numbers, the above results still hold.

For a number of series, we can also give the correspoding theorem, here not say any longer.

References

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正规发散级数

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摘 要

本文推广了维数正规发散的古典概念,将幂级数的一个定理推广到很一般的情况,然后讨论其在各种复函数项级数中的应用.