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Semitopological Systems*

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Abstract The aim of this paper is to establish the theory of semitopological systems, which has general topological spaces, fuzzy topological spaces, topological molecular lattices, and topological systems as special cases.

Keywords semitopological systems, spatializations and localizations, categorical equivalences

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1. Introduction

Neighbourhoods and open subsets are the most basic concepts in topology. The distinction between them is that an open subset must be neighbourhoods of all points which it includes, however that neighbourhoods don't so. Indeed, these neighbourhoods of a point can arbitrarily be defined without any restriction. Sierpinski, in [1], introduced the concept of neighbourhood spaces, that is, Fréchet(V) spaces, just in this way. The neighbourhood space is very general in the case of spaces. Prof. Wang Guojun, in [2, 3], researched two kinds of such neighbourhood spaces, called O-semitopological space and δ-semitopological space respectively. In Wang's semitopological spaces, however, those neighbourhoods are not open subsets in general. Hear, we introduce a kind of neighbourhood spaces, called S-spaces, in which the neighbourhood is neighbourhoods of all points which it includes, that is, the neighbourhood is open. It also belongs a type of semitopological spaces. Clearly every topological space is an S-space. Certainly one can discuss properties of S-spaces, which remains in other articles. Hear with a background of S-spaces, we introduce the concept of semitopological systems, and establish this theory.

Up to now, the study of topology is divided into two sects: having point and having no point. The former's field includes general topology, fuzzy topology, and topological molecular lattice theory; and the latter's field includes mainly locale theory. But, what we need to note is that these points in fuzzy topology and in topological molecular lattice theory fully differ from ones in general topology. The relation between point and set is logical relation instead of belongingness. Concretely, the relation between point and open subset depends on the quasi-coincidence relation [4], or that between point and closed subset(element) depends on remote-neighbourhood relation [5]. In locale theory, one also introduces the concept of

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points. the relation between point and open element differs from the cases above. All that enlighten one to generalize topological space so as to unify objects which the having point topology deals with. Vickers, in [6], introduced the concept of topological systems. We, in [7], showed that topological system can unite these objects: general topological space, fuzzy topological space, topological molecular lattice, and the spatialization of locale. Therefore the theory of topological systems should be concerned.

In this paper, the author introduces the concept of semitopological systems, which has S-spaces and topological systems (certainly, general topological spaces, fuzzy topological spaces, topological molecular lattices) as special cases. The aim of this paper is to establish the categorical theory of semitopological systems. The author, in [8], showed an application of semitopological systems in Domain Theory of denotational semantics of computer programming languages, and mainly established the Stone duality of L-domains with respect to stable functions by using the semitopological systems.

2 Semitopological system category

Definition 2.1 *Let X be a nonempty set, P a P-set, that is, P is a poset with the least element 0, and $\models$ a subset of the cartesian product $X \times P$. If $(x, a) \in \models$, the we call that x satisfies a, and denote $x \models a$. If $\models$ satisfies the following two statements:*

- $\forall x \in X$, $a, b \in P$, *if* $x \models a$, *and* $a \leq b$, *then* $x \models b$;
- $\forall x \in X$, $x \not\models 0$,

then we call $(X, P, \models)$ as a semitopological system, elements of X as points, and ones of P as open elements. We shall use notations $N, M, \ldots$ to represent semitopological systems, pt N and ΩN to do the point set and open elements set of N respectively.

Example 1 Every S-space is a semitopological system in which one can substitute $\models$ with $\in$.

Example 2 Suppose that X is a set of all programmes written in some programming language, which put out finite streams consisting of 0 and 1. Let $l \in X$. starts l represents the predicate which has l as a prefix. Define starts $l \leq$ starts m if and only if l has m as a prefix. Hence $\leq$ is an order-relation of the set $P = \{\text{starts } l : l \in X\}$, and starts $\varnothing$ is the least element, where $\varnothing$ is the empty stream. So P is a P-set.

$\forall x \in X$, $\forall$ starts $l \in P$, define $x \models$ starts l if and only if x puts out stream having l as a prefix. Then $(X, P, \models)$ is a semitopological system.

Definition 2.2 *Let N be a semitopological system. If ΩN is a inf-semilattice, and $\models$ satisfies the following statement:*

- *for each nonempty finite subset A of ΩN, $x \models \wedge A$ if and only if $x \models a$ for all $a \in A$, then we call N as an M-semitopological system.*

Definition 2.3 *Let N be an M-semitopological system. If ΩN is a frame, and $\models$ also satisfies:*

- *for each subset B of ΩN, $x \models \bigvee B$ if and only if there exists a $b \in B$ with $x \models b$,*

then we call N as a topological system [6].

The author, in [7], showed that both fuzzy topological spaces and topological molecular lattices are topological systems, and further semitopological systems. References [6, 7] deal with topological system theory and its application in Domain Theory. We also discuss semitopological systems' application in StableDomain Theory [8]. This paper will mainly deal with the categorical theory of semitopological systems.

Definition 2.4 *Suppose that N and M are semitopological systems. The continuous function f from N to M is a pair of functions $(\mathrm{pt} f, \Omega f)$, where*

- $\mathrm{pt} f: \mathrm{pt} N \to \mathrm{pt} M$ *is a mapping;*
- $\Omega f: \Omega M \to \Omega N$ *is a P-homorphism, that is, it is a monotone mapping preserving the least element;*
- $\forall x \in \mathrm{pt} N$, $\forall b \in \Omega M$, $\mathrm{pt} f(x) \models b$ *if and only if* $x \models \Omega f(b)$.

Further, if $\mathrm{pt} f$ is a bijection, and Ωf is an isomorphism, then we call continuous function f as a homeomorphism. If there exists a homeomorphism from N to M, then we say that semitopological systems N and M are homeomorphic. The identity on semitopological system N is $1_N = (1_{\mathrm{pt} N}, 1_{\Omega N})$.

We call two continuous functions f and G identical and denote $f = g$, if $\mathrm{pt} f = \mathrm{pt} g$, $\Omega f = \Omega g$.

We define the composition $g \circ f$ of continuous functions f and g as

$$\mathrm{pt}(g \circ f) = \mathrm{pt}\, g \circ \mathrm{pt} f, \quad \Omega(g \circ f) = \Omega f \circ \Omega g.$$

Clearly $g \circ f$ is continuous, and $g \circ 1_N = 3D\, 1_M \circ g$. So semitopological systems and continuous functions combine a category, called as semitopological system category, and denoted as **STS**.

Proposition 2.5 *Assume that N, M are S-spaces, and that $f = (\mathrm{pt} f, \Omega f)$ is a continuous function from N to M under the case of semitopological systems. Hence $\Omega f = (\mathrm{pt} f)^{-1}$.*

3 Spatialization of semitopological system

Definition 3.1 *Let N be a semitopological system. $\forall a \in \Omega N$, define*

$$\mathrm{ext}(a) = \{x \in \mathrm{pt} N : x \models a\}.$$

And set

$$\mathrm{ext}(\Omega N) = \{\mathrm{ext}(a) : a \in \Omega N\}.$$

Then $(\mathrm{pt} N, \mathrm{ext}(\Omega N))$ is an S-space, called the spatialization of semitopological system N, denoted as $\mathrm{Spat}\, N$.

By Definition 2.1, we know that $ext: \Omega N \to ext(\Omega N)$ is a P-homorphism, and that $x \vDash a$ if and only if $x \in ext(a)$. Then we get a natural continuous function $e = (1_{ptN}, ext)$ from Spat N to N.

Lemma 3.2 *Suppose that X is an S-space, and that N is a semitopological system, that f is a continuous function from X to N. Then there exists an unique continuous function* $\hat{f}: X \to$ Spat N *such that* $f = e \circ \hat{f}$.

Proof Assume that such $\hat{f}$ exists. Hence $f = e \circ \hat{f}$. And $pt f = pt e \circ pt\hat{f} = pt\hat{f}$, $\Omega f = \Omega\hat{f} \circ \Omega e = \Omega\hat{f} \circ ext$. Therefore the $\hat{f}$ could only be defined as:

$$pt\hat{f} = pt f, \quad \Omega = \hat{f}(ext(a)) = \Omega f(a).$$

As a result, $\hat{f}$ is unique. The remain is to verify that $\hat{f}$ is continuous.

At first, $\Omega\hat{f}$ is a P-homorphism. Let $ext(a) \subseteq ext(b)$, and $x \in \Omega\hat{f}(a)$. Then $pt f(x) \vDash a$ follows from the continuity of f. And further, $pt f(x) \in ext(a) \subseteq ext(b)$. So $pt f(x) \vDash b$, and $x \in \Omega f(b)$. As a result, $\Omega\hat{f}(a) \subseteq \Omega(b)$. That is, $\Omega\hat{f}$ preserves the order. Since $\Omega\hat{f}(\varnothing) = \Omega\hat{f}(ext(0)) = \Omega f(0) = \varnothing$, $\Omega\hat{f}$ is a P-homorphism.

In the end, taking $x \in ptX$, $ext(a) \in \Omega$Spat$N$, then

$$pt\hat{f}(x) \in ext(a) \Leftrightarrow pt f(x) \vDash a \Leftrightarrow x \in \Omega f(a) \Leftrightarrow x \in \Omega\hat{f}(ext(a)).$$

This has showed that $\hat{f}$ is a continuous function from Spat$N$ to M.

Notation **SPS** denotes the category of S-spaces which is a full subcategory of **STS**.

Theorem 3.3 *Fuctor* Spat: **STS**$\to$**SPS** *is a right adjoint functor of the inclusion functor I, that is*, I $\dashv$ Spat.

Proof Clearly, by Definition 3.1, for each $N \in obj$(**STS**), we have that Spat$N \in obj$(**SPS**). On the other hand, take $N, M \in obj$(**STS**), and a continuous function $f: N \to M$. Define Spat$f$: Spat$N \to$ Spat$M$ in the way:

$$pt\, \mathrm{Spat} f = pt f, \quad \Omega \mathrm{Spat} f(ext(a)) = ext(\Omega f(a)).$$

Then Spat$f$ is a continuous function, and indeed a morphism in the category **STS** by Proposition 2.5. Since

$$pt\mathrm{Spat}(f \circ g) = pt\mathrm{Spat} f \circ pt\mathrm{Spat} g,$$
$$\Omega\mathrm{Spat}(f \circ g) = \Omega\mathrm{Spat} g \circ \Omega\mathrm{Spat} f,$$

Spat is a functor from **STS** to **SPS**.

By Lemma 3.2, we get that $\forall N \in obj$(**STS**), $\forall X \in obj$(**SPS**), there exists a bijection between $Hom_{SPS}(X, \mathrm{Spat}N)$ and $Hom_{STS}(I(X), N)$. Therefore Spat is the right adjoint of I.

Definition 3.4 *If semitopological system N is homeomorphic to some S-space X, then we call semitopological system N spatial.*

Lemma 3.5 *Semitopological system N is spatial if and only if the natural mapping e:* Spat$N \to N$ *is a homeomorphism.*

Proof Only to show the necessity. Suppose X is an S-space and $f: X \to \hat{N}$ is a homeomorphism. Then by Lemma 3.2, there exists an unique continuous function $f: X \to \mathrm{Spat}N$ such that $f = e \circ f$. Since $\mathrm{pt}e$ is the identity mapping, the remain is to show that Ωe is an isomorphism. It is obvious that Ωe is surjective and that $\Omega f = \Omega f \circ \Omega e$, and that Ωf is an isomorphism. So, Ωe is injective, and further Ωe is a bijection and preserves the order. Since $(\Omega e)^{-1} = (\Omega f)^{-1} \circ \Omega f$, $(\Omega e)^{-1}$ is the order-preserving yet. As a result, Ωe is an isomorphism.

4 Localization of semitopological system

Definition 4.1 *Let A be a P-set, $\Theta = \{0, 1\}$. P-homorphisms from A to Θ are called points of A. We use the notation $\mathrm{pt}A$ to denote the set of all points of A. Taking $p \in \mathrm{pt}A$, $a \in A$, define $p \vDash a$ if and only if $p(a) = 1$. Then $(\mathrm{pt}A, A, \vDash)$ is a semitopological system, called as the localization of A, and denoted as $\mathrm{Loc}A$.*

Proposition 4.2 (1). *Suppose that A, B are P-sets, and that $f: B \to A$ is a P-homorphism. Then $\mathrm{Loc}f = (\mathrm{pt}f, f)$ is a continuous function from $\mathrm{Loc}A$ to $\mathrm{Loc}B$, where $\mathrm{pt}f(p) = p \circ f$. $\forall p \in \mathrm{pt}A$;*

(2) *Suppose that A, B, C are P-sets, and that $f: C \to B$, $g: B \to A$ are P-homorphisms. Then $\mathrm{Loc}(g \circ f) = \mathrm{Loc}g \circ \mathrm{Loc}f$.*

Definition 4.3 *Let N be a semitopological system. $\mathrm{Loc}\Omega N$ is called as the localization of N, and denoted as $\mathrm{Loc}N$.*

For each semitopological system N, there exists a natural mapping $P: N \to \mathrm{Loc}N$, where $x \in \mathrm{pt}N$, $\mathrm{pt}P(x) = 3D\,1$ if and only if $x \vDash b$, and $\Omega P = 1_{\Omega N}$.

Proposition 4.4 *Suppose that N, M are semitopological systems, and that $f: N \to M$ is a continuous function. Then $\mathrm{Loc}f = (\mathrm{pt}\Omega f, \Omega f)$ is a continuous function from $\mathrm{Loc}N$ to $\mathrm{Loc}M$, and the following graph is commutative:*

$$\begin{array}{ccccc} & N & \xrightarrow{P_N} & \mathrm{Loc}N & \\ f & \downarrow & & \downarrow & \mathrm{Loc}f \\ & M & \xrightarrow{P_M} & \mathrm{Loc}M & \end{array}$$

Proof It follows that $\mathrm{Loc}f$ is a continuous function from Proposition 4.2. $\forall x \in \mathrm{pt}N$, $\forall b \in \Omega N$, we get

$$\begin{aligned}(\mathrm{pt}P_M \circ \mathrm{pt}f)(x)(b) = 1 &\Leftrightarrow \mathrm{pt}P_M(\mathrm{pt}f(x))(b) = 1 \\ &\Leftrightarrow \mathrm{pt}f(x) \vDash b \\ &\Leftrightarrow x \vDash \Omega f(b),\end{aligned}$$

On the other hand,

$$\begin{aligned}(\mathrm{pt}\Omega f \circ \mathrm{pt}P_N)(x)(b) = 1 &\Leftrightarrow \mathrm{pt}\Omega f(\mathrm{pt}P_N f(x))(b) = 1 \\ &\Leftrightarrow \mathrm{pt}P_N(x) \vDash \Omega f(b) \\ &\Leftrightarrow x \vDash \Omega P_N(\Omega f(b)) \\ &\Leftrightarrow x \vDash \Omega f(b)\end{aligned}$$

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Therefore $\mathrm{pt}P_M \circ \mathrm{pt}f = \mathrm{pt}\Omega f \circ \mathrm{pt}P_N$, and further $\mathrm{pt}(P_M \circ f) = \mathrm{pt}(\mathrm{Loc}f \circ P_N)$. Since $\Omega(P_M \circ f) = \Omega f \circ \Omega P_M = \Omega(\mathrm{Loc}f \circ P_N$. We obtain that $P_M \circ f = \mathrm{Loc}f \circ P_N$.

We use the notation $\mathbf{P}_\perp$ to denote the category of P-sets and P-homorphisms, and F to denote the forgetful functor from the category **STS** to one $\mathbf{P}_\perp^{op}$. Then

Theorem 4.5 *F is the left adjoint of* Loc: $\mathbf{P}_\perp^{op} \to$ **STS**

Proof It follows that Loc is a functor from $\mathbf{P}_\perp^{op} \dashrightarrow$ **STS** from Proposition 2.4. Take $N \in obj$ (**STS**), $A \in obj(\mathbf{P}_\perp^{op})$, f is a continuous function from N to Loc$A$. Then Ωf is the unique P-homorphism from A to $F(N)$ which make the following graph commutative:

$$\begin{array}{ccccc} & F(N) & N \xrightarrow{P_N} & \mathrm{Loc} \circ F(N) & \\ \Omega f & \downarrow_{op} & \searrow & \downarrow & \mathrm{Loc}\Omega f = (\mathrm{pt}\Omega f, \Omega f) \\ & A & & \mathrm{Loc}A & \end{array}$$

that is, the arrow $P_N: N \to \mathrm{Loc} \circ F(N)$ is the universal arrow from N with respect to G. So F is the left adjoint of Loc.

Definition 4.6 *Semitopological system N is called as localic if it is homeomorphic to the localization* Loc$A$ *of some P-set A.*

Lemma 4.7 *Semitopological system N is localic if and only if the natural mapping* P: N $\to$ *L* ocN *is a homeomorphism.*

Proof Only do we show the necessity. Suppose that A is a P-set, that N is homeomorphic to Loc$A$, and that f is such homeomorphism. Then $\mathrm{pt}f: \mathrm{pt}N \to \mathrm{pt}A$ is a bijection, and $\Omega f: A \to \Omega N$ is an isomorphism. By Theorem 4.5, we get $f = \mathrm{Loc}\Omega f \circ P$. Hence

$$\mathrm{pt}f = \mathrm{pt}\Omega f \circ \mathrm{pt}P, \quad \Omega P = 1_{\Omega N}.$$

Since $\Omega P = 1_{\Omega N}$ is an isomorphism, the remain is to verify that $\mathrm{pt}P$ is a bijection.

For $\mathrm{pt}f$ is bijective, $\mathrm{pt}P$ is injective. Simultaneously, $\forall x \in \mathrm{pt}\,\Omega N$, $\mathrm{pt}\Omega f(x) \in \mathrm{pt}A$. There exists a $y \in \mathrm{pt}N$ such that $\mathrm{pt}f(y) = \mathrm{pt}\Omega f(x)$. Takeing arbitrarily $b \in \Omega N$, by the reason that Ωf is an isomorphism, we get an $a \in A$ with $\Omega f(a) = b$. Thus

$$\begin{aligned} \mathrm{pt}P(y) \vDash b \quad & \Leftrightarrow y \vDash \Omega P(b) \\ & \Leftrightarrow y \vDash b \\ & \Leftrightarrow y \vDash \Omega f(a) \\ & \Leftrightarrow \mathrm{pt}\Omega f(x) \vDash a \\ & \Leftrightarrow x \vDash \Omega f(a) \\ & \Leftrightarrow x \vDash b, \end{aligned}$$

that is, $\mathrm{pt}P(y) \vDash b \Leftrightarrow x \vDash b$. Hence $x = \mathrm{pt}P(y)$ from Proposition 4.5. So $\mathrm{pt}P$ is a surjection. As a result, P is a homeomorphism.

5 Specialization and categorical equivalence

Definition 5.1 *Let N be a semitopological system, and* $x, y \in \mathrm{pt}N$. *We call* $x \subseteq y$ *if for each* $a \in \Omega N$, $x \vDash$ *implies* $y \vDash a$. *We call the order-relation* $\subseteq$ *as specialization order.*

$x \subseteq y$ means that y possesses all those properties which x possesses So y includes more information than x. Thus the specialization order hasn't the 0antisymmetry: $x \subseteq y$ and $y \subseteq x$ imply $x = y$.

Definition 5.2 *Semitopological system N is called as T_0 if the specialization order of N satisfies the antisymmetry, that is, $x = y$ if and only if $x \subseteq y$ and $y \subseteq x$.*

Clearly, if N is a T_0 semitopological system, then $(\mathrm{pt}N, \subseteq)$ is a poset.

Proposition 5.3 *T_0 seperability dosen't change under homeomorphisms.*

Prposition 5.4 *Suppose that A is a P-set. Then* $\mathrm{Loc}A$ *is T_0, and further, for each semitopological system N,* $\mathrm{Loc}N$ *is T_0.*

Proof Take $x, y \in \mathrm{pt}A$. Then

$$\begin{aligned} x = y &\Leftrightarrow \forall b \in A,\ x(b) = y(b) \\ &\Leftrightarrow \forall b \in A,\ x \vDash b \Leftrightarrow y \vDash b \\ &\Leftrightarrow x \subseteq y,\ y \subseteq x. \end{aligned}$$

Proposition 5.5 *Suppose that N is a semitopological system, and that $P: N \to \mathrm{Loc}N$ is the natural mapping. Then $\mathrm{pt}P: \mathrm{pt}N \to \mathrm{pt}\Omega N$ preserves the specialization order.*

Lemma 5.6 *Suppose that N is a localic semitopological system, and that $P: N \to \mathrm{Loc}N$ is the natural mapping. Then $\mathrm{pt}P: \mathrm{pt}N \to \mathrm{pt}\Omega N$ is an isomorphism, where $\mathrm{pt}N$ is with the specialization order, and $\mathrm{pt}\Omega N$ is with the pointwise order $\leq$ of mappings, which also is the specialization order $\subseteq$ of $\mathrm{pt}\Omega N$.*

Proof It follows that $\mathrm{pt}P$ preserves the order, and is a bijection from Lemma 4.7. Therefore we only need to show that $(\mathrm{pt}P)^{-1}$ preserves

Take $x, y \in \mathrm{pt}\Omega N$, with $x \leq y$. Then $\forall b \in \Omega N$,

$$\begin{aligned} (\mathrm{pt}P)^{-1}(x) \vDash b &\Rightarrow (\mathrm{pt}P)^{-1}(x) \vDash \Omega P(b) \\ &\Rightarrow x \vDash b \\ &\Rightarrow y \vDash b \\ &\Rightarrow (\mathrm{pt}P)^{-1}(y) \vDash \Omega P(b) \\ &\Rightarrow (\mathrm{pt}P)^{-1}(y) \vDash b. \end{aligned}$$

So, $(\mathrm{pt}P)^{-1}(x) \subseteq (\mathrm{pt}P)^{-1}(y)$.

Theorem 5.7 *The following categories are dually equivalent:*

- *localic semitopological systems + continuous functions;*
- *P-sets + P-homorphisms, that is,* $\mathbf{P}_\perp$.

Proof From Lemma 4.7, for each localic semitopoligiacl system N, $\eta_N = 3D P: N \to \mathrm{Loc} \circ F(N) = \mathrm{Loc}N$ is an isomorphism. So η is a natural isomorphism from the identiti functor of the first category to functor $\mathrm{Loc} \circ F$. On the other hand, $\forall A \in obj(\mathbf{P}_\perp^{op})$,

$$\epsilon_A = 1_A: F \circ \mathrm{Loc}A = A \to A$$

is an isomorphism. So $\epsilon: F \circ \mathrm{Loc} \to 1_{\mathbf{P}_\perp^{op}}$ is a natural mapping. As a result, $(F, \mathrm{Loc}, \eta, \epsilon)$ *is an isomorphism adjoint.*

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References

1 Siperpinski W. *General topology*. Toronto, 1952.
2 Wang Guojun. *Semitopological spaces* (I). Journal of Shaanxi Normal University (natural), 1977, **5**: 34-54
3 Wang Guojun. *Semitopological spaces* (II). Journal of Shaanxi Normal University (natural), 1978, **6**: 10-34
4 Pu Baoming & Liu Yingming. *Fuzzy topology* I: *Neighbourhood structure of a fuzzy point and Moore-Smith convergence*. J. of Math. Anal. Appl., 1980, **76**: 577-599
5 Wang Guojun. *Theor of L-fuzzy topological spaces*. Press of Shaanxi Normal University, Xi'an, 1988.
6 Vickers S. *Topology via logic*. Cambridge University Press, London, 1989.
7 Chen Yixiang. *Topological system category and subtopological system*. Journal of Shaanxi Normal University, 1994, **22**(4): 19-23
8 Chen Yixiang. *Stone duality and representation of stable domains*. Journal of Computer and Mathematics with Applications, 1997, **34**(1): 27-41
9 Chen Yixiang. *Stable domain theory and its Stone representation theorem*. Sichuan University Ph. D thesis, 1995.
10 MacLane S. *Categories for the working mathematician*. World Publishing Corporation, Beijing, 1997.

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摘　要

本文旨在建立 S 系统理论，它具有相当的广泛性，以及应用背景性，同时又以拓扑空间、模糊拓扑空间、拓扑分子格以及拓扑系统为特例.