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Finite Groups whose Abelian Subgroup Orders Are Consecutive Integers*

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Abstract In this paper we give a complete classification of finite groups whose proper abelian subgroup orders are consecutive integers.

Keywords Sylow's theorem, solvable group, Frobenius group.

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In [1] the finite groups all of whose element orders are consecutive integers have been classified. Shi Wujie[2] has determined all finite groups whose proper subgroup orders are consecutive integers. In [2], Shi Wujie posed the following problem: To classify all groups in which proper abelian subgroup orders are consecutive integers. In this paper we answer the above question and obtain the following conclusion:

Theorem *Let G be a finite group and $\pi_a(G)$ the set of proper abelian subgroup orders in G. If $\pi_a(G)=\{1,2,\ldots,n\}$, then $n\le 6$ and one of the following holds.*

(1) $n=1$ *and* $G\cong Z_p$, *p prime.*
(2) $n=2$ *and* $G\cong Z_2\times Z_2$, *or* $G\cong Z_4$.
(3) $n=3$ *and* $G\cong S_3$, *or* $G\cong Z_6$.
(4) $n=4$ *and* $G\cong A_4$, *or* $G\cong S_4$.
(5) $n=5$ *and* $G\cong A_5$.
(6) $n=6$ *and* $G\cong S_5$.

All groups discussed will be assumed to be finite and all our notation is standard as can be found in Gorenstein[3]. Moreover, $A<G$ indicates that A is a proper subgroup of G and P_q denotes a q-Sylow subgroup of G, where q is a prime. For convenience, we call a group G whose proper abelian subgroup orders are consecutive integers an OA_n group, where n is the maximal integers in $\pi_a(G)$.

Lemma 1 *Let n be a positive integer and p_i the i-th prime of the prime series, $p_k=\max\{p_j \mid p_j\le n\}$. If $n\ge 5$, then following inequalities hold.*

(1) $2p_{k-1}+1>p_k$. (2) $p_{k-1}^2>n$.

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Proof We can immediately check (1) and (2) for $n \leq 20$. For $n > 20$ we have $p_{k-1} > (p_k - 1)/2$ by [2, Lemma 1] and (1) follows. Again by [2, Lemma 1] we see that

$$p_k > n/2$$

and

$$p_{k-1}^2 > ((p_k - 1)/2)^2 > n^2/16 - n/4 \geq n$$

and (2) follows.

Lemma 2 *Let G be a 2-group and $|G| \geq 16$. Then G has an abelian subgroup of oder 8.*

Proof Let A be a maximal abelian normal subgroup of G. Then $A = C_G(A)$ from [3, p. 185].

If $|A| = 4$, then

$$|\mathrm{Aut}(A)| = (2^2 - 1)(2^2 - 2) \text{ or } 2(2 - 1)$$

and

$$|G/A| = |G/C_G(A)| \,\big|\, |\mathrm{Aut}(A)|,$$

we have

$$|G| = |A| \text{ or } |G| = 2|A| = 4 \text{ or } 8,$$

a contradiction.

Lemma 3 *Let G be an OA_n group. Then G is solvable if and only if $n \leq 4$.*

Proof If $n \leq 4$, then G is obviously solvable.

If an OA_n group G is solvable and $n > 4$, by Lemma 1 we can suppose that $|G| = 2^{\alpha_1} 3^{\alpha_2} \ldots p_{k-2}^{\alpha_{k-2}} p_{k-1} p_k$ because a group of order p^2 (p prime) is abelian. Since G is solvable, there exists a subgroup H of G such that $|H| = p_{k-1} p_k$. By Sylow's theorem and Lemma 1, H is cyclic and $|H| = p_{k-1} p_k > p_{k-1}^2 > n$. This is a contradiction against the fact that G is an OA_n group.

Theorem 1 *Let G be an OA_n group where $n \leq 4$. Then one of following holds:*

(1) *$n = 1$ and $G \cong Z_p$, p prime.*

(2) *$n = 2$ and $G \cong Z_2 \times Z_2$, or $G \cong Z_4$.*

(3) *$n = 3$ and $G \cong S_3$, or $G \cong Z_6$.*

(4) *$n = 4$ and $G \cong A_4$, or $G \cong S_4$.*

Proof The conclusions (1) and (2) are obvious. For $n = 3$ we have $|G| = 6$ and (3) holds. For $n = 4$, Lemma 2 implies that $|G| = 12$ or 24. First, we shall shows that $P_3 \ntriangleleft G$.

Assume that $P_3 \triangleleft G$. Since G is an OA_4 group, obviously $C_G(P_3) = P_3$. As $|\mathrm{Aut}(P_3)| = 2$, we see that $|G/P_3| = 1$ or 2 and hence $|G| = 3$ or 6. It is plainly impossible.

Considering the minimal normal subgroup N of G, we have $|N| = 2$ or 4. If $|N| = 2$, then NP_3 is cyclic and $|NP_3| = 6$, a contradiction. Thus $|N| = 4$ and $|NP_3| = 12$. As NP_3 can not have any element of order 6, we infer that $NP_3 \cong A_4$.

If $|G| = 12$, then it follows that

$$G = NP_3 \cong A_4.$$

If $|G| = 24$, it is clear that $|G:NP_3| = 2$ and hence $NP_3 \lhd G$. It follows that $G \cong S_4$.

Theorem 2 *Let G be an OA_n group and $n > 4$. Then*:

(1) *We have $n \leq 6$.*

(2) $n = 5$, *then* $G \cong A_5$ *and if* $n = 6$, *then* $G \cong S_5$.

Proof We first deal with the case where $n = 5$ and 6. If G exists, then by Lemma 2 and Lemma 3 we can suppose that $|G| = 2^k \cdot 3 \cdot 5$ $(k < 4)$ and G is nonsolvable.

Let S be the maximal solvable normal subgroup of G and N/S a minimal normal subgroup of G/S. Then N/S is a nonabelian simple group and

$$\pi(N/S) = \{2, 3, 5\}.$$

Therefore S is 2-group and $S = O_2(G)$. From [4, p. 12] we have $N/S \cong A_5$. Also, G/S is isomorphic to a subgroup of $\mathrm{Aut}(N/S)$. And $G/S \cong A_5$ or S_5.

By Lemma 2 it is clear that

$$|S| = |O_2(G)| = 1 \text{ or } 2.$$

If $|O_2(G)| = 2$, then $O_2(G)P_5$ is a cyclic group of order 10. Thus

$$O_2(G) = S = 1$$

and

$$G \cong A_5 \text{ or } S_5.$$

(2) follows.

Let G be an OA_n group, where $n \geq 7$,

$$|G| = 2^{\alpha_1}3^{\alpha_2}\ldots p_j^{\alpha_j}p_{j+1}\ldots p_k, \quad 2p_{j+1} > n \geq 2p_j,$$

where p_i is the ith prime of the prime series. As 9 does not divide the order of $SL(2, 5)$, the proof of [1, Lemma 7] implies that G is neither Frobenius nor 2-Frobenius and a proof similar to [1, Lemma 8] yields that G has a normal series which contains a nonabelian simple factor group G_1 such that $p_{j+1}, \ldots, p_k \in \pi(G_1)$.

In the case where $7 \leq n \leq 10$, the above description implies that G has a nonabelian simple factor group $\overline{G}_1 = G_1/S$ such that $5, 7 \in \pi(G_1)$. According to the first part of the proof of [1, Lemma 9], G_1 can only be isomorphic to A_7 or $\mathrm{PSL}(3, 4)$. Since G is an OA_n group where $7 \leq n \leq 10$, we can suppose that $|G| = 2^{\alpha_1}3^{\alpha_2} \cdot 5 \cdot 7$ and so $\pi(S) \subseteq \{2, 3\}$. Because S has a fixed-point-free automorphism of order 7, S is a nilpotent $\{2, 3\}$-group.

Assume that 3 divides the order of S. Considering the 3-Sylow subgroup Q_3 of S, we have $Q_3 \lhd G_1$ because S is nilpotent. Obviously $|Z(Q_3)| = 3$ or 9. By Sylow's theorem $Z(Q_3)P_7$ is abelian and $|Z(Q_3)P_7| \geq 21$ where P_7 denotes the 7-Sylow subgroup of G_1, a contradiction. Thus S is a 2-group and $S = O_2(G_1)$.

As $Z(O_2(G_1))$ is abelian, it is clear that $|Z(O_2(G_1))| \leq 8$. When $|Z(O_2(G_1))| = 2$ or 4, from Sylow's theorem we can easily prove that $Z(O_2(G_1))P_7$ is abelian and

$$|Z(O_2(G_1))P_7| \geq 14,$$

a contradiction. When $|Z(O_2(G_1))| = 8$, we similarly have $Z(O_2(G_1))P_5$ is abelian and

$$|Z(O_2(G_1))P_5| = 40,$$

where P_5 denotes the 5-Sylow subgroup of G_1, a contradiction. Thus $|Z(O_2(G_1))| = 1$ and $O_2(G_1) = S = 1$. Therefore G_1 is a nonabelian normal simple subgroup of G and $G_1 \cong A_7$ or $PSL(3, 4)$.

As $G_1 \lhd G$ and $C_G(G_1) = 1$ we infer that

$$A_7 \leq G \leq S_7 \text{ or } PSL(3,4) \leq G \leq Aut(PSL(3,4)).$$

When $A_7 \leq G \leq S_7$, we conclude that

$$G \cong A_7 \text{ or } S_7.$$

But both A_7 and S_7 are not OA_n groups because A_7 contains an abelian subgroup of order 9 and does not contain any abelian subgroup of order 8 and S_7 contains elements of order 12. When $PSL(3,4) \leq G \leq Aut(PSL(3,4))$, G is not an OA_n group because $PSL(3,4)$ has a subgroup which is isomorphic to $Z_4 \times Z_4$. Therefore there does not exist any OA_n group with $7 \leq n \leq 10$.

For $n \geq 11$ we can also prove that there does not exist any OA_n group using an argument similar to [1, Lemma 10].

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Abel 子群的阶为连续整数的有限群

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摘　　要

本文给出了所有有限群它的Abel子群的阶为连续整数的分类