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Monotone Iterative Technique on Boundary Value Problems for Second Order Impulsive Integrodifferential Equations of Mixed Type in Banach Spaces*

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Abstract In this paper, the author uses the monotone iterative technique and cone theory to investigate the extremal solutions of two-point boundary value problems for nonlinear second order impulsive integrodifferential equations of mixed type in Banach speces based on a comparison result.

Keywords Monotone iterative technique, boundary value problem, impulsive integrodifferential equation.

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1. Introduction

The theory of impulsive differential equations is a new important branch of differential equations (see [1]). Consider the existence of extremal solutions of two point boundary value problems (BVP) for nonlinear second order impulsive integrodifferential equations of mixed type in Banach space E.

$$\begin{cases} -u''=f(t,u,Tu,Su), & t\neq t_k, \\ \triangle u|_{t=t_k}=I_k(u(t_k)), & \\ \triangle u'|_{t=t_k}=\bar{I}_k(u(t_k)), & k=1,2,\ldots,m, \\ au(0)-bu'(0)=x_0, & cu(1)+du'(1)=x_1, \end{cases} \tag{1}$$

where $f\in C[J\times E\times E\times E, E]$, $J=[0,1]$, E is a real Banach space, $0<t_1<\ldots<t_m<1$, I_k, $\bar{I}_k\in C[E,E]$, $a\geq 0$, $b\geq 0$, $c\geq 0$, $d\geq 0$, $ac+ad+bc>0$ are constant, x_0, $x_1\in E$, $\triangle u|_{t=t_k}=u(t_k^+)-u(t_k^-)$, where $u(t_k^+)$ and $u(t_k^-)$ denote the right and left limits of $u(t)$ at $t=t_k$, respectively, $k=1,2,\ldots,m$. $\triangle u'|_{t=t_k}$ has a similar meaning for $u'(t)$, and the operators T, S are given by

$$Tu(t)=\int_0^t k(t,s)u(s)ds, \qquad Su(t)=\int_0^1 k_1(t,s)u(s)ds, \tag{2}$$

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with $k \in C[D, R^+]$, $k_1 \in C[D_0, R^+]$, $D = \{(t, s) \in R^2 \mid 0 \le s \le t \le 1\}$, $D_0 = \{(t, s) \in R^2 \mid 0 \le t, s \le 1\}$, $R^+ = [0, +\infty]$. In the special case where $T = S = \theta$, [2] discussed the existence of solutions of BVP (1) by means of the fixed point theory. In this paper, we shall obtain a comparion result for the BVP (1) and prove an existence theorem of minimal and maximal solutions of BVP (1) by means of the monotone iterative technique and cone theory.

2 Several Lemmas

Let $PC^1[J, E] = \{u: u$ is a map from J into E such that $u(t)$ is continuously differentiable at $t \ne t_k$, left continuous at $t = t_k$ and $u(t_k^+)$, $u'(t_k^-)$, $u'(t_k^+)$ exist, $k = 1, 2, \ldots, m\}$. Evidently, $PC^1[J, E]$ is a Banach space with norm $\|u\|_{PC^1} = \max\{\|u\|_{PC}, \|u'\|_{PC}\}$ where $\|u\|_{PC} = \sup_{t \in J} \|u(t)\|$, $\|u'\|_{PC} = \sup_{t \in J} \|u'(t)\|$. Notice that $PC[J, E] = \{u: u$ is a map from J into E such that $u(t)$ is continuous at $t \ne t_k$, left continuous at $t = t_k$, and $u(t_k^+)$ exists, $k = 1, 2, \ldots, m\}$ is also a Banach space with norm $\|u\|_{PC} = \sup_{t \in J} \|u(t)\|$. Let the Banach space E be partially ordered by a cone P of E; i.e., $u \le v$ if and only if $v - u \in P$. P is said to be normal if there exists a positive constant N such that $\theta \le u \le v$ implies $\|u\| \le N\|v\|$. Let $K = \{u \in PC[J, E]: u(t) \ge \theta$ for all $t \in J\}$. Then K is a cone in space $PC[J, E]$, and so $PC[J, E]$ is partially ordered by K: $u \le v$ iff $v - u \in K$; i.e., $u(t) \le v(t)$ for all $t \in J$. Evidently, if P is normal, then K is also normal. The properties of the cone and the partial order may be found in [3]. Let $J' \triangleq J \setminus \{t_1, t_2, \ldots, t_m\}$. A map $u \in PC^1[J, E] \cap C^2[J', E]$ is called a solution of BVP (1) if it satisfies (1).

Consider the BVP

$$\begin{cases} -u'' + M^2 u = g(t), & t \ne t_k, \\ \triangle u|_{t=t_k} = I_k(u(t_k)), & \\ \triangle u'|_{t=t_k} = \bar{I}_k(u(t_k)), & k = 1, 2, \ldots, m, \\ au(0) - bu'(0) = x_0, & cu(1) + du'(1) = x_1, \end{cases} \tag{3}$$

where $M > 0$ is a constant, $g \in PC[J, E]$. For convenience, we denote $I_k = I_k(u(t_k))$, $\bar{I}_k = \bar{I}_k(u(t_k))$.

Lemma 1 *$u \in PC^1[J, E] \cap C^2[J', E]$ is a solution of BVP (3) if and only if $u \in PC[J, E]$ is a solution of the following impulsive integral equation.*

$$u(t) = H_0(t)x_0 + H_1(t)x_1 + \int_0^1 G(t, s)g(s)ds + \sum_{k=1}^{m} [G(t, t_k)(-\bar{I}_k) + H(t, t_k)I_k], \tag{4}$$

where

$$G(t, s) = \frac{\rho}{2M} \begin{cases} H_0(t)H_1(s), & 0 \le s < t \le 1, \\ H_0(s)H_1(t), & 0 \le t \le s < 1, \end{cases} \tag{5}$$

$$H(t, s) = \frac{\rho}{2M} \begin{cases} H_0(t)H_1'(s), & 0 \le s < t \le 1, \\ H_0'(s)H_1(t), & 0 \le t \le s < 1, \end{cases} \tag{6}$$

$$H_0(t) = \frac{1}{\rho}\left((dM + c)e^{M(1-t)} + (dM - c)e^{-M(1-t)}\right), \tag{7}$$

$$H_1(t) = \frac{1}{\rho}\left((bM + a)e^{Mt} + (bM - a)e^{-Mt}\right), \tag{8}$$

$$\rho = e^{M}(dM + c)(bM + a) - (dM - c)(bM - a)e^{-M} > 0. \tag{9}$$

Proof First suppose that $u \in PC^1[J, E] \cap C^2[J', E]$ is a solution of BVP (3), and let $D = d/dt$. Then

$$-(D - M)(D + M)u = g(t), \quad t \neq t_k.$$

Let $y(t) = (D + M)u(t)$. Then

$$(e^{Mt}u(t))' = e^{Mt}y(t), \qquad (D - M)y(t) = -g(t), \qquad t \neq t_k. \tag{10}$$

It is easy to see by integration of (10) that

$$u(t) = e^{-Mt}\left(u(0) + \int y(s)e^{Ms}ds + \sum_{0<t_k<t} e^{Mt_k} I_k\right), \quad t \in J, \tag{11}$$

$$y(t) = e^{Mt}\left(u'(0) + Mu(0) - \int g(s)e^{-Ms}ds + \sum_{0<t_k<t} e^{-Mt_k}(\bar{I}_k + MI_k)\right), \quad t \in J. \tag{12}$$

Substituting (12) into (11), by means of $au(0) - bu'(0) = x_0$, $cu(1) + du'(1) = x_1$, and some computations, we can obtain (4).

Conversely, assume that $u \in PC[J, E]$ is a solution of Equation (4). Direct differentiation on (4) implies, $u(t) \in PC^1[J, E] \cap C^2[J', E]$ is a solution of Equation (3).

Consider the linear BVP

$$\begin{cases} -u'' + M^2u = -NTu - N_1Su + h(t), & t \neq t_k, \\ \triangle u|_{t=t_k} = I_k(\eta(t_k)), \\ \triangle u'|_{t=t_k} = \bar{I}_k(\eta(t_k)), & k = 1, 2, \ldots, m, \\ au(0) - bu'(0) = x_0, & cu(1) + du'(1) = x_1, \end{cases} \tag{13}$$

where $M > 0$, $N \geq 0$, $N_1 \geq 0$ are constant, $h, \eta \in PC[J, E]$. In the following denote

$$k^* = \max_{(t,s)\in D} |k(t, s)|, \quad k_1^* = \max_{(t,s)\in D_0} |k(t, s)|.$$

Lemma 2 If

$$Nk^* + N_1k_1^* < M^2, \tag{14}$$

then the linear BVP (13) has exactly one solution $u \in PC^1[J, E] \cap C^2[J', E]$ *given by*

$$u(t) = H_0^*(t)x_0 + H_1^*(t)x_1 + \int G^*(t, s)h(s)ds + \sum_{k=1}^{m}[G^*(t, t_k)(-\bar{I}_k) + H^*(t, t_k)I_k]. \tag{15}$$

Where

$$H_0^*(t) = H_0(t) + \int_0^1 Q(t, r)H_0(r)dr, \quad H_1^*(t) = H_1(t) + \int_0^1 Q(t, r)H_1(r)dr, \quad (16)$$

$$G^*(t, s) = G(t, s) + \int_0^1 Q(t, r)G(r, s)dr, \quad H^*(t, s) = H(t, s) + \int_0^1 Q(t, r)H(r, s)dr, \quad (17)$$

$$Q(t, s) = \sum_{n=1}^{\infty} k_2^{(n)}(t, s),$$

$$k_2^{(n)}(t, s) = \int_0^1 \cdots \int_0^1 k_2(t, r_1)\, k_2(r_1, r_2) \cdots k_2(r_{n-1}, s)\,dr_1 \cdots dr_{n-1},$$

$$k_2(t, s) = -N\int_0^1 G(t, r)k(r, s)dr - N_1\int_0^1 G(t, r)\,k_1(r, s)dr,$$

Moreover the following iequlities hold:

$$|k_2(t, s)| \le \frac{1}{M^2}(Nk^* + N_1k_1^*) = k_2^*, \ (t, s) \in D_0, \qquad |Q(t, s)| \le \frac{k_2^*}{(1 - k_2^*)}, \ t, s \in J.$$

The proof is similar to that of the lemma in [4]. We omit it.

Lemma 3 *If*

$$Nk^* + N_1k_1^* \le \min\{l_1, l_2\}, \quad (18)$$

where

$$l_1 = \frac{1}{4bdM}\left(-\rho + \sqrt{\rho^2 + 16(bdM)^2}\right), \ l_2 = \frac{M}{2}\left(-(e^M - 1) + \sqrt{(e^M - 1)^2 + 4M^2}\right),$$

is satisfied, then (14) *is satisfied and*

$$H_0^*(t) \ge 0, \ H_1^*(t) \ge 0, \ G^*(t, s) \ge 0, \ \forall (t, s) \in D_0,$$

where $H_0^*(t)$, $H_1^*(t)$ *and* $G^*(t, s)$ *are given by* (16) *and* (17).

The proof is analogous to that of the Corollary 1 in [4].

Corollary 1 (Comparison Result) *Let* (18) *be satisfied, suppose that* $u \in PC^1[J, E] \cap C^2[J', E]$ *satisfies*

$$\begin{cases} -u'' + M^2u \ge -NTu - N_1Su, & t \ne t_k, \\ \triangle u|_{t=t_k} = \theta, \\ \triangle u'|_{t=t_k} \le \theta, & k = 1, 2, \ldots, m, \\ au(0) - bu'(0) \ge \theta, & cu(1) + du'(1) \ge \theta, \end{cases}$$

then $u(t) \ge \theta$ *for* $t \in J$.

Let $J_0 = [0, t_1]$, $J_1 = (t_1, t_2]$, ..., $J_{m-1} = (t_{m-1}, t_m]$, $J_m = (t_m, 1]$. For $B \subset PC[J, E]$, we denote $B(t) = \{u(t): u \in B\} \subset E$ $(t \in J)$.

Lemma 4[5] *If* B *is bounded and the elements of* B *are equicontinuous on each* J_k $(k = 0, 1, \ldots, m)$, *then* $\alpha(B) = \sup_{t \in J}\alpha(B(t))$, *where* α *denotes the Kuratowski measure of noncompact-*

ness.

3 Existence theorem

In the following, we shall use the ordered interval $[p, q]$ in space $PC[J, E]$, i.e.,

$$[p, q] = \{u \in PC[J, E]: p \leq u \leq q,$$

i.e.,

$$p(t) \leq u(t) \leq q(t) \text{ for all } t \in J\}.$$

Let us list some assumptions:

(H_1) there are $p, q \in PC^1[J, E] \cap C^2[J', E]$, $p(t) \leq q(t)$ for $t \in J$ such that

$$\begin{cases} -p'' \leq f(t, p, Tp, Sp), \quad t \neq t_k, \\ \triangle p|_{t=t_k} = I_k(p(t_k)), \\ \triangle p'|_{t=t_k} \geq \bar{I}_k(p(t_k)), \quad k = 1, 2, \ldots, m, \\ ap(0) - bp'(0) \leq x_0, \quad cp(1) + dp'(1) \leq x_1; \end{cases}$$

$$\begin{cases} -q'' \geq f(t, q, Tq, Sq), \quad t \neq t_k, \\ \triangle q|_{t=t_k} = I_k(q(t_k)), \\ \triangle q'|_{t=t_k} \leq \bar{I}_k(q(t_k)), \quad k = 1, 2, \ldots, m, \\ aq(0) - bq'(0) \geq x_0, \quad cq(1) + dq'(1) \geq x_1. \end{cases}$$

(H_2) there exist $M > 0$, $N \geq 0$ and $N_1 \geq 0$ such that

$$f(t, u, v, w) - f(t, \bar{u}, \bar{v}, \bar{w}) \geq -M^2(u - \bar{u}) - N(v - \bar{v}) - N_1(w - \bar{w}),$$

whenever

$$t \in J,\ p(t) \leq \bar{u} \leq u \leq q(t),\ Tp(t) \leq \bar{v} \leq v \leq Tq(t),\ Sp(t) \leq \bar{w} \leq w \leq Sq(t).$$

(H_3) $I_k(u) = x_0^{(k)}$, whenever $p(t_k) \leq u \leq q(t_k)$, where $x_0^{(k)}$ is a fixed element of E, $\bar{I}_k(u) \leq \bar{I}_k(\bar{u})$ whenever $p(t_k) \leq \bar{u} \leq u \leq q(t_k)$, $k = 1, 2, \ldots, m$.

(H_4) $Nk^* + N_1 k_1^* \leq \min\{l_1, l_2\}$, where l_1, l_2 are given by Lemma 3.

Theorem *Let $P \subset E$ be a regular cone, (H_1), (H_2), (H_3) and (H_4) be satisfied, then there exist sequences $\{p_n(t)\}, \{q_n(t)\} \subset PC^1[J, E] \cap C^2[J', E]$ such that*

$$p(t) = p_0(t) \leq p_1(t) \leq \ldots \leq p_n(t) \leq \ldots \leq q_n(t) \ldots \leq q_1(t) \leq q_0(t) = q(t). \quad (19)$$

and $p_n(t) \to u_*(t)$, $q_n(t) \to u^*(t)$ as $n \to \infty$ uniformly in t, $u_*, u^* \in PC^1[J, E] \cap C^2[J', E]$. Moreover, u_* and u^* are minimal and maximal solutions of BVP (1) on the ordered interval $[p, q]$ respectively.

Proof For any $\eta \in [p, q]$ (i.e., $\eta \in PC[J, E]$ and $p(t) \leq \eta(t) \leq q(t)$ for $t \in J$). Consider linear BVP (13) where

$$h(t) = f(t, \eta(t), T\eta(t), S\eta(t)) + M^2\eta(t) + NT\eta(t) + N_1S\eta(t) \tag{20}$$

By lemma 2, (13) has exactly one solution $u \in PC^1[J, E] \cap C^2[J', E]$ given by (15). Define $A\eta = u$, then A is an operator from $[p, q]$ into $PC^1[J, E] \cap C^2[J', E] \subset PC[J, E]$ and η is a solution of BVP (1) if and only if $\eta = A\eta$.

By Corollary 1, we can show

$$p \le Ap, \qquad Aq \le q \tag{21}$$

and

$$A\eta_1 \le A\eta_2, \quad \text{if} \quad p \le \eta_1 \le \eta_2 \le q. \tag{22}$$

Now, let $p_0 = p$, $q_0 = q$, $p_n = Ap_{n-1}$, $q_n = Aq_{n-1}$ $(n = 1, 2, \ldots)$. It follows from (21) and (22) that (19) holds. By definition of A, we have

$$p_n(t) = H_0^*(t)x_0 + H_1^*(t)x_1 + \int G^*(t, s)h_{n-1}(s)ds + \sum_{k=1}^{m}[G^*(t, t_k)(-\bar{I}_k(p_{n-1}(t_k)) + H^*(t, t_k)I_k(p_{n-1}(t_k))], \tag{23}$$

where

$$h_{n-1}(t) = f(t, p_{n-1}(t), Tp_{n-1}(t), Sp_{n-1}(t)) + M^2p_{n-1}(t) + NTp_{n-1}(t) + N_1Sp_{n-1}(t).$$

Finally, we shall show that $p_n(t) \to u_*(t)$, $q_n(t) \to u^*(t)$ $(n \to \infty)$ uniformly in t, u_*, $u^* \in PC^1[J, E] \cap C^2[J', E]$ are minimal and maximal solutions of BVP (1) on the ordered interval $[p, q]$ respectively.

Let $B = \{p_n\} \subset [p, q]$, $B(t) = \{p_n(t)\} \subset E$, $t \in J$. In the following, we shall show B is a relatively compact. By virtue of the regularity of P and (19), we first have $B(t)$ is a relatively compact, i.e.,

$$\alpha(B(t)) = 0, \qquad t \in J \tag{24}$$

and then, we obtain that K is a normal cone in $PC[J, E]$. Therefore, $[p, q]$ is bounded set in $PC[J, E]$. By the normality of K, we can prove $\{p_n'(t) | n \in N\}$ is a bounded set in $PC[J, E]$. Applying the mean value theorem, we obtain that the elements of $B = \{p_n\}$ are equicontinuous on each J_k, $k = 1, 2, \ldots, m$. By Lemma 4 and (24), we have $\alpha(B) = \sup_{t \in J} \alpha(B(t)) = 0$. Hence, $\{p_n\}$ is a relatively compact set in $PC[J, E]$. In view of the normality of K and (19), $\{p_n\}$ in $PC[J, E]$ converges to $u_* \in [p, q]$, i.e., $\{p_n(t)\}$ converges to $u_*(t)$ uniformly on J. Taking limits in (23) as $n \to \infty$, we get that u_* is a solution of BVP (1).

Similarly, we can show that $\{q_n\}$ in $PC[J, E]$ converges to $u^* \in PC[J, E]$ and that u^* is a solution of BVP (1). It follows by using standard arguments in [3] that u_*, u^* are minimal and maximal solutions BVP (1) on the ordered interval $[p, q]$ respectively.

Remark By the theorem 2.2 in [6], if E is weakly complete and P is normal, then P is regular. Hence, the main result in our paper for the case that E is weakly complete and P is normal still holds.

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Banach 空间中二阶混合型脉冲积微分方程边值问题的单调迭代技巧

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摘　　要

利用单调迭代技巧和锥理论研究了 Banach 空间中二阶混合型脉冲积微分方程的两点边值问题的极解.