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A Class of Oscillatory Singular Integrals *

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Abstract L^p mapping properties are considered for a class of oscillatory signular integral operators.

Keywords Calderón-Zygmund kernel. oscillatory singular integral operator. polynomial growth estimate.

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1. Introduction

We will work on $\mathbf{R}^n$ ($n \geq 1$). Let $K(x)$ be a standard Calderón-Zygmund kernel, i.e., K is C^1 away from the origin, has mean value zero on each sphere centered at the origin, and for some positive constant A,

$$|K(x)| \leq A|x|^{-n}, \quad |\nabla K(x)| \leq A|x|^{-n-1}. \tag{1}$$

Let $\Phi(x) \in C^\infty(\mathbf{R}^n \backslash \{0\})$ be a real-valued function which satisfies

$$|D^\alpha \Phi(x)| \leq B|x|^{a-|\alpha|}, \text{ for } |\alpha| \leq 3, \tag{2}$$

$$\sum_{|\alpha|=2} |D^\alpha \Phi(x)| \geq B'|x|^{a-2}, \tag{3}$$

where a is a fixed real number, B and B' are positive constants. Define the oscillatory singular integral operator

$$Tf(x) = \text{p.v.} \int_{\mathbf{R}^n} e^{i\Phi(x-y)} K(x-y) f(y) \mathrm{d}y. \tag{4}$$

For the special case $\Phi(x) = |x|^a$, such operators have been studied by many authors (see [1 – 4], for example). Recently, Fan and Pan [5] considerded the operator defined by (4) with phase function Φ satisfying (2) and (3), and established the L^p ($1 < p < \infty$) and H^1 boundedness for this operator.

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The purpose of this paper is to consider the L^p mapping properties for a class of oscillatory singular integral operators related to the operators defined by (4). Let m be a positive integer, $a \in L^\infty(\mathbf{R}^n)$. The operators we consider here are of the form

$$T_{a,m}f(x) = \int_{\mathbf{R}^n} e^{i\Phi(x-y)}K(x-y)(V_{x,y}a)^m f(y)\mathrm{d}y, \tag{5}$$

where K is a Calderón-Zygmund kernel and

$$V_{x,y}a = \int_0^1 a(x+t(y-x))\mathrm{d}t.$$

For the case of $n=1$, if we set $A(x) = \int_0^x a(t)\mathrm{d}t$, it is easy to verify that

$$K(x-y)(V_{x,y}a)^m = K(x-y)\left(\frac{A(x)-A(y)}{x-y}\right)^m$$

also satisfies (1) and the corresponding singular integral operator is bounded on $L^p(\mathbf{R}^n)$ for $1<p<\infty$. Thus by repeating the arguement used in [5], we see that in this case, the operator defined by (5) is bounded on $L^p(\mathbf{R}^n)$ for $1<p<\infty$. On the other hand, for the case of $n \geq 2$, $K(x-y)(V_{x,y}a)^m$, the kernel of the operator defined (5), fails to satisfy the well-known Hömander condition required in the classical Calderón-Zygmund theory. Christ and Journé [6] showed that the L^p bound of the $n-$commutator defined by

$$\tilde{T}_{a,m}f(x) = \int_{\mathbf{R}^n} K(x-y)(V_{x,y}a)^m f(y)\mathrm{d}y, \tag{6}$$

satisfies a polynomial growth estimate, i.e., for each $\mu > 2$ and $1<p<\infty$, there exists a constant $C = C_{\mu,p}$, such that

$$||\tilde{T}_{a,m}f||_p \leq Cm^\mu ||a||_\infty^m ||f||_p.$$

It is natural to conjecture that the L^p bound of the operator $T_{a,m}$ also satisfies the same estimate. In this paper, we will prove that this is true. Our result may be stated as follows.

Theorem *Let $1<p<\infty$, T be defined as in (4). Suppose that Φ satisfies (2) and (3) for some $a \neq 0$. Then for each $\mu > 2$, there exists a positive constant $C = C_{\mu,p}$ such that*

$$||T_{a,m}f||_p \leq Cm^\mu ||a||_\infty^m ||f||_p.$$

2. Proof of Theorem

We begin with a preliminary lemma which will be used in the proof of our Theorem.

Lemma 1 [7] *Let $K(x,y)$ be a distribution which away from the diagonal $\{x=y\}$ agrees with a function satisfing*

$$|K(x,y)| \leq A|x-y|^{-n}.$$

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Let $1 < p < \infty$. Suppose that the operator

$$Tf(x) = \int_{\mathbf{R}^n} K(x,y)f(y)\mathrm{d}y$$

is bounded on $L^p(\mathbf{R}^n)$ with bound $\|T\|$. Then the truncated operator

$$T_\varepsilon f(x) = \int_{|x-y|\le\varepsilon} K(x,y)f(y)\mathrm{d}y$$

is bounded on $L^p(\mathbf{R}^n)$ with bound $C(\|T\| + A)$, and C is independent of ε.

Proof of Theorem Without loss of generality, we may assume that $\|a\|_\infty = 1$. Let $\mu > 2$. Let $\varphi \in C_0^\infty(\mathbf{R}^n)$ such that

$$\operatorname{supp}\varphi \subset \{1/2 \le |x| \le 2\} \text{ and } \sum_{j=-\infty}^{\infty} \varphi(2^{-j}x) \equiv 1, \text{ for } |x| \ne 0.$$

Set $\varphi_j(x) = \varphi(2^{-j}x)$ for integer j. To prove our Theorem, we consider the following two cases.

Case I. $a > 0$. Let $\psi(x) = 1 - \sum_{j=1}^\infty \varphi_j(x)$. It is obvious that $\operatorname{supp}\psi \subset \{|x| \le 4\}$ and $\psi(x) \equiv 1$ if $|x| < 1$. Write

$$\begin{aligned} T_{a,m}f(x) &= \int_{\mathbf{R}^n} e^{i\Phi(x-y)}\psi(x-y)K(x-y)(V_{x,y}a)^m f(y)\mathrm{d}y + \\ &\quad \sum_{j=1}^\infty \int_{\mathbf{R}^n} e^{i\Phi(x-y)}\varphi_j(x-y)K(x-y)(V_{x,y}a)^m f(y)\mathrm{d}y \\ &= T_{a,m}^0 f(x) + \sum_{j=1}^\infty T_{a,m}^j f(x). \end{aligned}$$

Let us consider the term $T_{a,m}^0$ first. Set

$$\begin{aligned} |T_{a,m}^0 f(x)| &= \left|\int_{|x-y|<1} \psi(x-y)K(x-y)(V_{x,y}a)^m f(y)\mathrm{d}y\right| + \\ &\quad \left|\int_{|x-y|<1} \left(e^{i\Phi(x-y)} - 1\right)\psi(x-y)K(x-y)(V_{x,y}a)^m f(y)\mathrm{d}y\right| + \\ &\quad \left|\int_{|x-y|\ge1} e^{i\Phi(x-y)}\psi(x-y)K(x-y)(V_{x,y}a)^m f(y)\mathrm{d}y\right| \\ &= \mathrm{E} + \mathrm{F} + \mathrm{G}. \end{aligned}$$

Recall that $\psi(x) \equiv 1$ for $|x| < 1$. Thus

$$\mathrm{E} = \left|\int_{|x-y|<1} K(x-y)(V_{x,y}a)^m f(y)\mathrm{d}y\right|.$$

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The result of Christ and Journé [6] and Lemma 1 shows that

$$\|\mathrm{E}\|_p \le Cm^{\mu}\|f\|_p,\ \ 1<p<\infty.$$

On the other hand, by the fact $a>0$ and (2), trivial computation shows that

$$\mathrm{F} \le C\int_{|x-y|\le 1} |x-y|^{-n+a}|f(y)|\mathrm{d}y \le CMf(x),$$

where Mf is the Hardy-Littlewood maximal function of f. So we have

$$\|\mathrm{F}\|_p \le C\|f\|_p,\ \ 1<p<\infty.$$

It is obvious that $\mathrm{G}\le CMf(x)$. Therefore

$$\|\mathrm{G}\|_p \le C\|f\|_p,\ \ 1<p<\infty.$$

Combining the estimates for E, F and G yields that

$$\|T^0_{a,m}f\|_p \le Cm^{\mu}\|f\|_p,\ \ 1<p<\infty.$$

Now we turn our attention to the operator $T^j_{a,m}$ $(j\ge 1)$. We have the following crude estimate

$$\|T^j_{a,m}f\|_p \le C\|f\|_p,\ \ 1<p<\infty. \tag{7}$$

We want to obtain a refined L^2 estimate for $T^j_{a,m}$, i.e., we want to show that there exist a positive constant δ such that

$$\|T^j_{a,m}f\|_2 \le Cm2^{-\delta j}\|f\|_2, \tag{8}$$

If we can do this, interpolation between inequalities (7) and (8) then gives that

$$\|T^j_{a,m}f\|_p \le Cm2^{-\tilde{\delta} j}\|f\|_p,\ 1<p<\infty, \tag{9}$$

with $\tilde{\delta}>0$. Summing over the last inequality for all $j\ge 1$ leads to our desired estimate.

The proof of Theorem is now reduced to proving (8). Set

$$U^j_{a,m}f(x)=\int_{\mathbf{R}^n} e^{i\Phi(2^j(x-y))}\varphi(x-y)K(x-y)(V_{x,y}a)^m f(y)\mathrm{d}y. \tag{10}$$

By dilation-invariance, we see that the inequality (8) is equivalent to the estimate

$$\|U^j_{a,m}f\|_2 \le Cm2^{-\delta j}\|f\|_2. \tag{11}$$

Write $\mathbf{R}^n=\cup_d Q_d$, where each Q_d is a cube having side length 1, and these cubes $\{Q_d\}$ have disjoint interiors. Set $f_d=f\chi_d$. Since that the support of $U^j_{a,m}f_d$ is contained in a fixed multiple of Q_d, thus the supports of various terms $\{U^j_{a,m}f_d\}_d$ have bounded overlaps and

$$\|U^j_{a,m}f\|_2^2 \le C\sum_d \|U^j_{a,m}f_d\|_2^2.$$

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So we may assume that supp $f \subset Q$ for some cube Q having side length 1. Let $\eta(x, y) \in C_0^\infty(\mathbf{R}^n \times \mathbf{R}^n)$, η is identically one on $50nQ \times 50nQ$ and vanishes outside $100nQ \times 100nQ$. Set $F(x, y) = (V_{x,y}a)^m \eta(x, y)$. The fact that supp $f \subset Q$ implies that

$$U_{a,m}^j f(x) = \int_{\mathbf{R}^n} e^{\Phi(2^j(x-y))} \varphi(x-y) K(x-y) F(x, y) f(y) \mathrm{d}y.$$

Let ε be a small positive constant which will be chosen later. Write

$$F(x, y) = F_1(x, y) + F_2(x, y),$$

where

$$F_1(x, y) = \left(\chi_{|\xi|+|\eta|>2^{\varepsilon j}} \hat{F}(\xi, \eta)\right)^\vee (x, y),$$

$\wedge$ denotes the Fourier transform and $\vee$ denotes the inverse Fourier transform. Decompose the operator $U_{a,m}^j$ as

$$\begin{aligned} U_{a,m}^j f(x) &= \int_{\mathbf{R}^n} e^{i\Phi(2^j(x-y))} K(x-y) \varphi(x-y) F_1(x, y) f(y) \mathrm{d}y + \\ &\quad \int_{\mathbf{R}^n} e^{i\Phi(2^j(x-y))} K(x-y) \varphi(x-y) F_2(x, y) f(y) \mathrm{d}y \\ &= U_{a,m}^{j,\mathrm{I}} f(x) + U_{a,m}^{j,\mathrm{II}}(x). \end{aligned}$$

To estimate these two terms, we will use the following two lemmas.

Lemma 2 (see [8, page 402]) *There exists a positive constant $\alpha < 1/3$ such that*

$$\int\int_{\mathbf{R}^n \times \mathbf{R}^n} |\hat{F}(\xi, \eta)|^2 (1 + |\xi| + |\eta|)^{2\alpha} \mathrm{d}\xi \mathrm{d}\eta \leq C m^2.$$

Lemma 3 (see [5]) *Suppose that Φ satisfies (2) and (3). Then There exist a positive constant C such that for each $j \in \mathbf{Z}$, the operator*

$$U^j h(x) = \int_{\mathbf{R}^n} e^{i\Phi(2^j(x-y))} \varphi(x-y) K(x-y) h(y) \mathrm{d}y \tag{12}$$

satisfies

$$\|U^j h\|_2 \leq C 2^{-ja/2} \|h\|_2.$$

We now return to the proof of (11). By Schwarz's inequality, it is not difficult to find that

$$|U_{a,m}^{j,\mathrm{I}} f(x)|^2 \leq C \int_{\mathbf{R}^n} |F_1(x, y)|^2 \mathrm{d}y \|f\|_2^2.$$

Lemma 2 together with Plancherel's theorem then shows that

$$\begin{aligned} \|U_{a,m}^{j,\mathrm{I}} f(x)\|_2^2 &\leq C\|f\|_2^2 \int\int_{\mathbf{R}^n \times \mathbf{R}^n} |F_1(x,y)|^2 \mathrm{d}x \mathrm{d}y \\ &\leq C\|f\|_2^2 \int\int_{|\xi|+|\eta|>2^{\varepsilon j}} |\hat{F}_1(\xi, \eta)|^2 \mathrm{d}\xi \mathrm{d}\eta \\ &\leq C m^2 2^{-2\varepsilon\alpha j} \|f\|_2^2. \end{aligned}$$

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It remains to estimate $U^{j,\mathrm{II}}_{a,m}$. For each fixed $\eta \in \mathbf{R}^n$, set $f_\eta = e^{i\eta y} f(y)$. By Fubini's theorem, we have

$$U^{j,\mathrm{II}}_{a,m} f(x) = \int\int_{|\xi|+|\eta|\le 2^{\varepsilon j}} e^{ix\xi} \hat{F}(\xi, \eta) \mathrm{U}^j f_\eta(x) \mathrm{d}\xi \mathrm{d}\eta.$$

Thus by Minkowski's inequality and Lemma 2 and Lemma 3, it follows that

$$\begin{aligned} \|U^{j,\mathrm{II}}_{a,m} f\|_2 &\le C \int\int_{|\xi|+|\eta|\le 2^{\varepsilon j}} |\hat{F}(\xi, \eta)| \|U^j f_\eta\|_2 \mathrm{d}\xi \mathrm{d}\eta \\ &\le C 2^{-ja/2} 2^{\varepsilon n j} \left(\int\int_{\mathbf{R}^n \times \mathbf{R}^n} |\hat{F}(\xi, \eta)|^2 \mathrm{d}\xi \mathrm{d}\eta \right)^{1/2} \|f\|_2 \\ &\le C m 2^{-j(a/2-\varepsilon n)} \|f\|_2. \end{aligned}$$

Let $\varepsilon = a/4n$. Combining the estimates for $U^{j,\mathrm{I}}_{a,m}$ and $U^{j,\mathrm{II}}_{a,m}$ leads to the estimate (11).

Case II. $a < 0$. Let $\phi(x) = 1 - \sum_{j=-\infty}^{-1} \varphi_j(x)$. it is easy to see that $\operatorname{supp}\phi \subset \{|x| > 1/4\}$ and $\phi(x) \equiv 1$ if $|x| > 1$. Decompose $T_{a,m}$ as

$$\begin{aligned} T_{a,m} f(x) &= \int_{\mathbf{R}^n} e^{i\Phi(x-y)} \phi(x-y) K(x-y) (V_{x,y} a)^m f(y) \mathrm{d}y + \\ &\quad \sum_{j=-\infty}^{-1} \int_{\mathbf{R}^n} e^{i\Phi(x-y)} \varphi_j(x-y) K(x-y) (V_{x,y} a)^m f(y) \mathrm{d}y \\ &= \tilde{T}^0_{a,m} f(x) + \sum_{j=-\infty}^{-1} T^j_{a,m} f(x). \end{aligned}$$

As in the case I, we have that

$$\begin{aligned} |\tilde{T}^0_{a,m} f(x)| &\le \left| \int_{|x-y|>1} K(x-y) (V_{x,y} a)^m f(y) \mathrm{d}y \right| + \\ &\quad \int_{|x-y|>1} |x-y|^{-n+a} |f(y)| \mathrm{d}y + \int_{1/4\le|x-y|\le 1} |x-y|^{-n} |f(y)| \mathrm{d}y \\ &\le \left| \int_{|x-y|>1} K(x-y) (V_{x,y} a)^m f(y) \mathrm{d}y \right| + C M f(x), \end{aligned}$$

where in the last inequality, we haved used the fact that $a < 0$. Thus by the result of Christ and Journé [6] and Lemma 1, we obtain that

$$\|\tilde{T}^0_{a,m} f\|_p \le C m^\mu \|f\|_p, \quad 1 < p < \infty.$$

On the other hand, for $j \le -1$, by spliting $F(x, y)$ as

$$F(x, y) = \tilde{F}_1(x, y) + \tilde{F}_2(x, y)$$

with

$$\tilde{F}_1(x,y) = \left(\chi_{|\xi|+|\eta|>2^{-\varepsilon j}} \hat{F}(\xi, \eta) \right)^\vee (x, y),$$

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and repeating the arguement used in Case I, we can obtain that there exists some $\gamma < 0$, such that

$$\|T^j_{a,m} f\|_p \le Cm2^{-\gamma j}\|f\|_p, \; j \le -1, \; 1 < p < \infty.$$

This finishes the proof of Theorem for the case of $a < 0$.

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一类振荡奇异积分

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摘　要

考虑了一类振荡奇异积分算子 L^p 性质.

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