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A Remark on Ciarlet-Raviart Mixed Finite Element Method *

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Abstract In this paper, a variant of Ciarlet-Raviart mixed finite element scheme for solving the biharmonic equation is introduced, in which linear and quadratic elements are used for approximating the vorticity $-\Delta\phi$ and the stream function ϕ, respectively. Under the conditions that triangulation is quasi-uniform, it is proved that the scheme has the same order of accuracy as the standard Ciarlet-Raviart scheme using quadratic finite elements.

Keywords mixed finite element, Ciarlet-Raviart scheme.

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Consider the biharmonic problem

$$\begin{aligned} \Delta^2\phi &= g, \qquad G, \\ \phi = \frac{\partial\phi}{\partial r} &= 0, \qquad \partial G, \end{aligned} \tag{1}$$

where G is a convex polygonal domain in R^2. We denote by $\|*\|_s$ and $|*|_s$ the norm and seminorm of Sobolev space $H^s(G)$, respectively. $H^0(G)=L^2(G)$. It is well know that if g $\in H^{-1}(G)$, then there is a unique solution ϕ in $H^3(G)$. And the solution ϕ also satisfies

$$\|\phi\|_3 \le C\|g\|_{-1}. \tag{2}$$

For the two-dimensional steady-state flows problem, ϕ and $-\Delta\phi$ denote the stream function and the vorticity, respectively. Let $u=-\Delta\phi$, we obtain the variational formulation of (1): find $(u,\phi)\in H^1(G)\times H_0^1(G)$ satisfying

$$\begin{aligned} &\int_G uv\mathrm{d}x - \int_G \nabla v\nabla\phi\mathrm{d}x = 0, \quad \forall v\in H^1(G), \\ &\int_G \nabla u\nabla\psi\mathrm{d}x = \int_G g\psi\mathrm{d}x, \quad \forall\psi\in H_0^1(G), \end{aligned} \tag{3}$$

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Let T_h be a quasi-uniform triangulation of G by triangles e of diameter less than or equal to h,

$$X_h = \{v \in C(G) : v|_e \in P_k, \forall e \in T_h\},$$
$$M_h = X_h \cap H_0^1(G).$$

Ciarlet-Raviart scheme for approximating the solution (3) is defined by the determining of a pair $(u_h, \phi_h) \in X_h \times M_h$ such that

$$\int_G u_h v \mathrm{d}x - \int_G \nabla v \nabla \phi_h \mathrm{d}x = 0, \quad \forall v \in X_h,$$
$$\int_G \nabla u_h \nabla \psi \mathrm{d}x - \int_G g\psi \mathrm{d}x = 0, \quad \forall \psi \in M_h. \tag{4}$$

It is well-known (see [1], [2]) that the error estimates

$$\|\phi - \phi_h\|_1 \le Ch^{s-1}\|\phi\|_s,$$
$$\|u - u_h\|_\delta \le Ch^{s-2-\delta}\|\phi\|_s, \quad \delta = 0,1$$

hold whenever $1 \le s \le \min(k+1, r), \phi \in H^r(G)$.
The same space X_h is used by Ciarlet and Raviart for the approximation of both spaces $H^1(G)$ and $H_0^1(G)$. We note that the above error estimates dependent of degree k of the finite element space and regularity of the weak solution, and the weak solution $(u, \phi) \in H^1(G) \times H^3(G)$ only. In view of the fact, we are naturally to choose that the X_h and M_h are the linear and the quadratic finite element space, respectively, in the (4).

Condition 1 *Assume that $V_i \subset H^1(G)$ is the finite element space of degree i on a quasi-uniform partition T_{ih} with mesh size ih, $I_i : C(G^-) \to V_i$, (i=1,2), and*

$$I_2 I_1 = I_2, I_1 I_2 = I_1, |I_2 v - I_1 v|_1 \le \alpha |I_1 v|_1, \quad \forall v \in C(G^-), \tag{5}$$

where $0 < \alpha < 1$.

Let T_h bi obtained from T_{2h} by dividing each element into four congruent triangles. Gao,J.B. and the authors of the paper[5] prove that the condition 1 holds and $\alpha = \sqrt{\frac{2}{3}}$. Let $X_h = V_1, M_h = V_2 \cap H_0^1(G)$. This choice leads to a variant of Ciarlet-Raviart scheme. Our goal in this paper is to prove the efficiency of the variant.

Falk-Osbon's following results (see [2]) are fundamental to the analysis of this paper.

Let H, M and X be real Banach spaces with norms $\|*\|_H, \|*\|_M$ and $\|*\|_X$, respectively. Let $X \subset H$ with a continuous injection . Let $a(*,*)$ and $b(*,*)$ be continuous bilinear forms on $H \times H$ and $X \times M$, respectively.

$$|a(u,v)| \le C\|u\|_H \|v\|_H, \quad \forall u, v \in H, \tag{6}$$
$$|b(u,v)| \le C\|u\|_X \|v\|_M, \quad \forall u \in X, v \in M. \tag{7}$$

Consider abstract problem: Given $f \in X', g \in M'$, find $(u, \phi) \in X \times M$ such that

$$a(u,v) - b(v,\phi) = \langle f, v\rangle, \quad \forall v \in X,$$
$$b(u,\psi) = \langle g, \psi\rangle, \quad \forall \psi \in M. \tag{8}$$

where $\langle *, * \rangle$ denotes duality between X (or M) and its topological dual X' (or M'). The weak form of (8) is as follows: Find $(u_h, \phi_h) \in X_h \times M_h$, such that

$$
\begin{aligned}
&a(u_h, v) - b(v, \phi_h) = \langle f, v \rangle, \quad \forall v \in X_h, \\
&b(u_h, \psi) = \langle g, \psi \rangle, \quad \forall \psi \in M_h,
\end{aligned} \tag{9}
$$

where $X_h \subset X, M_h \subset M$ are both the finite dimensional space. Falk, R.S. proved (see [2]).

Lemma 1 *Assume that the following conditions are satisfied:*

H1) $\forall (f, g) \in D$, *the (8) has a unique solution, where D is subspace of $X' \times M'$.*

H2) Let E be Banach space, $M \subset E$ with a continuous injection, then $\forall d \in E'$, the dual problem: find a $(y_d, z_d) \in X \times M$ such that

$$
\begin{aligned}
a(v, y_d) - b(v, z_d) &= 0, \quad \forall v \in X, \\
b(y_d, \psi) - \langle d, \psi \rangle &= 0, \quad \forall \psi \in M
\end{aligned} \tag{10}
$$

has one and only one solution.

H3) There is a constant $r > 0$, independent of h such that

$$a(v, v) \geq r \|v\|_H^2, \quad \forall v \in X_h.$$

H4) There is constant $s(h)$ satisfying

$$\|v\|_X \leq s(h) \|v\|_H, \quad \forall v \in X_h.$$

H5) There is an operator $P : Y \to X_h$ satisfying

$$b(y - Py, \psi) = 0, \quad \forall y \in Y, \psi \in M_h,$$

where $Y = \mathrm{span}(\{y_d\}_{d \in E'}, u), (u, \phi)$ is a solution of the (8), and (y_d, z_d) is a solution of the (10).
Then the (9) has a unique solution (u_h, ϕ_h) satisfying the following error estimates

$$
\|u - u_h\|_H \leq C(\|u - Pu\|_H + s(h)\|\phi - \psi\|_M), \quad \forall \psi \in M_h, \tag{11}
$$

$$
\begin{aligned}
\|\phi - \phi_h\|_E \leq \; & \sup_d \frac{1}{\|d\|_{E'}} (b(y_d - Py_d, \phi - \psi) + \\
& a(u - u_h, Py_d - y_d) + b(u - u_h, z_d - v)), \quad \forall \psi, v \in M_h.
\end{aligned} \tag{12}
$$

We are now ready to apply Lemma 1 to analyze the error of the variant.

Theorem 1 *Assume that the Condition 1 is satisfied. Then the (4) has a unique solution (u_h, ϕ_h) satisfying the following error estimates*

$$\|u - u_h\|_0 \leq Ch, \tag{13}$$

$$\|\phi - \phi_h\|_\delta \leq Ch^2, \quad \delta = 0, 1. \tag{14}$$

Proof Choose $X = H^1(G), M = H_0^1(G), H = L^2(G)$, $a(u,v) = \int_G uv\mathrm{d}x$, $b(v,\psi) = \int_G \nabla v \nabla \psi \mathrm{d}x$, $D = O \times H^{-1}(G), E = H_0^1(G), E' = H^{-1}(G)$. The (3) and (4) thus have forms of the (8) and (9), respectively. It is obvious that the (6), (7), and H1)–H3) hold. Using the inverse assumption we have that H4 holds and $S(h) = Ch^{-1}$. In order to prove Theorem 1 we must check still condition H5) and estimate $\|u - Pu\|_0, \|y_d - Py_d\|_0$ and $|y_d - Py_d|_1$. For given $v \in H^1(G)$, consider an auxiliary problem: find $w \in V_1$ to satisfy

$$\int_G \nabla w \nabla \psi \mathrm{d}x = \int_G \nabla v \nabla \psi \mathrm{d}x, \quad \forall \psi \in V_2, \tag{15}$$

$$\int_G w \mathrm{d}x = \int_G v \mathrm{d}x. \tag{16}$$

Using Condition 1 , we know that (15) equivalent to

$$\int_G \nabla w \nabla I_2 \psi \mathrm{d}x = \int_G \nabla v \nabla I_2 \psi \mathrm{d}x, \quad \forall \psi \in V_1. \tag{17}$$

Consider the quotient space $H^1(G)/P_0$, where P_0 is the space of polynomials of degree 0. This space is a Banach space, when it is equipped with the quotient norm

$$\|v^0\|_1 = \inf_{p \in P_0} \|v + p\|_1,$$

where $v^0 \in H^1(G)/P_0$ denotes the equivalence class of the element $v \in H^1(G)$. Given $v \in v^0$ we have

$$|v^0|_1 = |v|_1, \tag{18}$$

$$\|v^0\|_1 \le C|v^0|_1. \tag{19}$$

Define on the $H^1(G)/P_0$

$$(u^0, v^0)_0 = \int_G \nabla u \nabla v, \quad \forall u^0, v^0 \in H^1(G)/P_0. \tag{20}$$

Since

$$(u^0, v^0)_0 = (v^0, u^0)_0, \tag{21}$$

$$(u^0 + w^0, v^0)_0 = (u^0, v^0)_0 + (w^0, v^0)_0, \tag{22}$$

$$(u^0, u^0)_0 = |u|_1^2 \ge C\|u^0\|_1^2. \tag{23}$$

Therefore the $(u^0, v^0)_0$ is an inner product on $H^1(G)/P_0$. $(u^0, u^0)_0^{\frac{1}{2}} = |u|_1$. Using definition of the $\|v^0\|_1$, we deduce

$$|v|_1 = \inf_{p \in P_0} |v + p|_1 \le \inf_{p \in P_0} \|v + p\|_1 = \|v^0\|_1. \tag{24}$$

From (19), (18) and (24), we know that $(u^0,u^0)_0^{\frac{1}{2}}$ equivalent to $\|u^0\|_1$. Therefore the $H^1(G)/P_0$ is also a Hilbert space with inner product $(u^0,v^0)_0$. Define the quotient space V_1/P_0, it is clear that the V_1/P_0 is a subspace of the $H^1(G)/P_0$. Let

$$I(w^0,v^0)=\int_G \nabla w\nabla(I_2 v)\mathrm{d}x. \tag{25}$$

From (5), $\forall w\in V_1$

$$\begin{aligned}\int_G \nabla w\nabla(I_2 w)\mathrm{d}x &= \int_G \nabla w\nabla(I_2 w - I_1 w + I_1 w)\mathrm{d}x\\ &= \int_G |\nabla w|^2 dx - \int_G \nabla w\nabla(I_1 w - I_2 w)\mathrm{d}x\\ &\geq |w|_1^2-\alpha|w|_1^2=(1-\alpha)|w|_1^2. \end{aligned}\tag{26}$$

Combining (25), (26), (18) and (19), we obtain

$$I(w^0,w^0)\geq C\|w^0\|_1^2,\quad \forall w^0\in V_1/P_0. \tag{27}$$

From (25), (5) and (24), we have

$$\begin{aligned}I(w^0,v^0) &= \int_G \nabla w\nabla(I_2 v)\mathrm{d}x \leq |w|_1|I_2 v|_1\\ &\leq C|w|_1|v|_1\leq C\|w^0\|_1\|v^0\|_1,\quad \forall w^0,v^0\in V_1/P_0. \end{aligned}\tag{28}$$

Then, note that $w^0+u^0=(w+u)^0$. We deduce $I(w^0,v^0)$ is a continuous, symmetric, positive definite bilinear form on $V_1/P_0\times V_1/P_0$. For given v, define

$$g(\psi^0)=\int_G \nabla v\nabla(I_2\psi)\mathrm{d}x. \tag{29}$$

It is clear that $g(\psi^0): V_1/P_0\to R$ is a continuous linear form. By Lax-Milgram Lemma we know that

$$\begin{aligned} w^0 &\in V_1/P_0,\\ I(w^0,\psi^0) &= g(\psi^0),\quad \forall\psi^0\in V_1/P_0 \end{aligned}\tag{30}$$

have unique solution w^0, and thus $\forall v\in H^1(G)$, the solution satisfying (16) of the (15) is unique. This defines an operator $P: H^1(G)\to V_1$ that satisfy H5). And thus there exits unique (u_h,ϕ_h) satisfing (4). In order to obtain (13) and (14) by using (11) and (12), let us to estimate $|Py_d-y_d|_1$ and $\|Py_d-y_d\|_0$. From $E'=H^{-1}(G)$ and (2), we know $z_d\in H^3(G)\cap H_0^2(G), y_d\in H^1(G)$ and

$$\|z_d\|_3\leq C\|d\|_{-1},\quad \|y_d\|_1\leq C\|d\|_{-1}. \tag{31}$$

It is easily seen that $Py_d \in V_1$

$$\int_G \nabla(Py_d - y_d)\nabla\psi \mathrm{d}x = 0, \quad \forall\psi \in V_2. \tag{32}$$

From (26), (32) and (5), we deduce that

$$\begin{aligned} |Py_d|_1^2 &\le C\int_G \nabla Py_d \nabla I_2 Py_d \mathrm{d}x = C\int_G \nabla y_d \nabla I_2 Py_d \mathrm{d}x \\ &\le C|y_d|_1 |I_2 Py_d|_1 \le C|y_d|_1 |Py_d|_1, \end{aligned}$$

and that

$$\begin{aligned} |Py_d|_1 &\le C|y_d|_1, \\ |Py_d - y_d|_1 &\le C|y_d|_1 \le C\|d\|_{-1}. \end{aligned} \tag{33}$$

Let us to derive an estimate for $\|y_d - Py_d\|_0$ by means of the well known Nitsche techniques. Let z satisfy

$$\begin{aligned} z &\in H^1(G), \\ b(v, z) &= (v, y_d - Py_d), \quad \forall v \in H^1(G). \end{aligned} \tag{34}$$

Using the regularity results (see [1]), we have that $z \in H^2(G)$ and

$$\|z\|_2 \le C\|y_d - Py_d\|_0. \tag{35}$$

Letting $v = y_d - Py_d$, and using (32), (33), (35) and interpolation error estimates, we obtain

$$\begin{aligned} \|y_d - Py_d\|_0^2 &= b(y_d - Py_d, z) = b(y_d - Py_d, z - I_2 z) \\ &\le |y_d - Py_d|_1 |z - I_2 z|_1 \le C\|d\|_{-1} Ch\|z\|_2 \\ &\le Ch\|d\|_{-1}\|y_d - Py_d\|_0, \end{aligned}$$

and that

$$\|y_d - Py_d\|_0 \le Ch\|d\|_{-1}. \tag{36}$$

Similarly

$$\|u - Pu\|_0 \le Ch. \tag{37}$$

From (11) and (37), we obtain

$$\|u - u_h\|_0 \le C(\|u - Pu\|_0 + Ch^{-1}\|\phi - I_2\phi\|_1) \le Ch.$$

The (13) is proved. Letting $v = I_2 z_d$ and $\psi = I_2\phi$ in the (12), from (33), (36), (13), (31) and interpolation error estimates, we deduce

$$\begin{aligned} \|\phi - \phi_h\|_1 &\le \sup_{d \in H^{-1}(G)} \frac{1}{\|d\|_{-1}}\big(b(y_d - Py_d, \phi - I_2\phi) + \\ &\qquad a(u - u_h, Py_d - y_d) + b(u - u_h, z_d - I_2 z_d)\big) \\ &\le \sup_{d \in H^{-1}(G)} \frac{C\|d\|_{-1} Ch^2\|\phi\|_3 + Ch^2\|d\|_{-1} + Ch^2\|d\|_{-1}}{\|d\|_{-1}} \\ &\le Ch^2. \end{aligned}$$

The (14) is proved. And the proof of the Theorem has been completed.

Remark Our method differs from the Ciarlet-Raviart method only in the choice of the finite dimensional space X_h.

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关于 Ciarlet-Raviart 混合有限元法的一个注记

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摘 要

本文推广解双调合方程的 Ciarlet-Raviart 混合有限元方案：用二次元逼近流函数 ϕ，一次元逼近涡度 $-\Delta\phi$. 在拟一致三角形剖分的条件下，证明了推广方案具有 ϕ 和 $-\Delta\phi$ 都用二次元逼近的标准 Ciarlet-Raviart 方案同样的精度阶.