Journal of Mathematical Research & Exposition
Vol.19, No.3, 475-482, August 1999

Extensions of Congruences on Subsemigroups of Semigroups *

LI Yong-hua[1], *ZHANG Mou-cheng*[2]

1. Dept. of Math., Zhongshan University, Guangzhou 510275;
1. Dept. of Math., South China Normal University, Guangzhou 510631;
2. Dept. of Math., South China Normal University, Guangzhou 510631

Abstract: Let ρ be a congruence on a semigroup S. ρ is called a rectangular band congruence if S/ρ is a rectangular band. In this paper the least rectangular band congruence on a semigroup is discribed. Let T be a subsemigroup of a semigroup S. A necessary and sufficient condition for which every rectangular band congruence on T can be uniquely extended to a rectangular band congruence on S is given.

Key words: semigroups; extension of congruence.

Classification: AMS(1991) 20M10/CLC O152.7

Document code: A **Article ID:** 1000-341X(1999)03-0475-08

1. Introduction

Let S be a semigroup and ρ be a congruence on S. We call ρ a left (right) zero congruence on S if S/ρ is a left (right) zero semigroup. We call ρ a rectangular band congruence on S if S/ρ is a rectangular band. Clearly, left (right) zero congruence is rectangular band congruence. Since the universal relation on a semigroup S is a rectangular band (left zero, right zero) congruence on S, always there exists the least rectangular band (left zero, right zero) congruence on S. We denote the least rectangular band congruence on a semigroup S by ρ^{rn}. To obtain all rectangular band congruences, it is sufficient to obtain ρ^{rn} since there is a lattice isomorphism between the rectangular band congruences on S and the cnogruences on S/ρ^{rn} (see [4]).Congruences on a rectangular band can be easily described (see [2]). Theorem 3.7 in [3] describes the least rectangular band congruence on an orthodox semigroup S . In this paper the least rectangular band congruence on a semigroup is characterized.

*Received date: 1996-09-24

Foundation item: Natural Science Foundation of Guangdong Province, China (970321)

Biography: LI Yonghua, male, born in Nanchang county, Jiangxi province. M.Sc, currently an associate professor at South China Normal University.

In general congruences on some subsemigroups of a semigroup S are easier to be described than ones on S. Mills in [3] considered the way extended the rectangular band congruence on the band $E(S)$ of all idempotents of an orthodox semigroup S to a rectangular band congruence on S. It is shown in [3] that every rectangular band congruence on the band $E(S)$ of all idempotents of a orthodox semigroup S can be extended uniquely to a rectangular band congruence on S. Furter,if the least rectangular band congruence on $E(S)$ can be extended to one on S, then every rectangular band congruence on $E(S)$ can also be extended to a rectangular band congruence on S. In the section 3 of this paper a necessary and sufficient condition for which every rectangular band congruence on a subsemigroup T of a semigroup S can be uniquely extended to a rectangular band congruence on S is given.

2. The Least Rectangular Band Congruence

Let S be a semigroup. We define ρ^l, ρ^r, ρ^m, $\rho^{m'}$ on S as follows:

(1) For all x, y in S, $x\rho^l y$ if and only if there exist $x_1, x_2, \cdots, x_n$ in S,where $n \in Z^+$, such that

$$xS \cap x_1S \neq \emptyset, x_1S \cap x_2S \neq \emptyset, \cdots, x_nS \cap yS \neq \emptyset.$$

(2) For all x, y in $S, x\rho^r y$ if and only if there exist $x_1, x_2, \cdots, x_n$ in S, where $n \in Z^+$, such that

$$Sx \cap Sx_1 \neq \emptyset, Sx_1 \cap Sx_2 \neq \emptyset, \cdots, Sx_n \cap Sy \neq \emptyset.$$

(3) For all x, y in S, $x\rho^m y$ if and only if there exist $x_1, y_1, x_2, y_2, \cdots, x_n, y_n$ in S, where $n \in Z^+$, such that

$$xSx \cap x_1Sy_1 \neq \emptyset, x_1Sy_1 \cap x_2Sy_2 \neq \emptyset, \cdots, x_nSy_n \cap ySy \neq \emptyset.$$

(4) $\rho^{m'} = \rho^l \cap \rho^r$.

Theorem 2.1 *Let S be a semigroup. Then ρ^l, ρ^r, $\rho^m(\rho^{m'})$ are the least left zero congruence, the least right zero congruence and the rectangular band congruence respectively, and $\rho^m = \rho^{m'}$.*

Proof Clearly ρ^l, ρ^r and ρ^m are equivalent relations on S.

Let $(a, b) \in \rho^l$. Then there exist $x_1, x_2, \cdots, x_n$ in S, such that

$$aS \cap x_1S \neq \emptyset, x_1S \cap x_2S \neq \emptyset, \cdots, x_nS \cap bS \neq \emptyset.$$

It is clear that for any $t \in S$,

$$taS \cap tx_1S \neq \emptyset, tx_1S \cap tx_2S \neq \emptyset, \cdots, tx_nS \cap tbS \neq \emptyset,$$

$$atS \cap aS \neq \emptyset, aS \cap x_1S \neq \emptyset, x_1S \cap x_2S \neq \emptyset, \ldots, x_nS \cap bS \neq \emptyset.$$

Thus $(ta, tb), (at, bt) \in \rho^l$. Therefore ρ^l is a congruence on S.

For all a, b in S, since $abS \cap aS \neq \emptyset, (ab, a) \in \rho^l$. Thus ρ^l is a left zero congruence on S.

If $(a,b)\in\rho^l$ then there exist x_i,u_j,v_j in S, where $i=1,2,\cdots,n, j=1,2,\cdots,n+1$, such that

$$au_1=x_1v_1, x_1u_2=x_2v_2,\cdots,x_nu_{n+1}=bv_{n+1}.$$

If ρ is a left zero congruence on S,then

$$a\rho=au_1\rho=x_1v_1\rho=x_1\rho=x_1u_2\rho=x_2v_2\rho=\cdots=x_nu_{n+1}\rho=bv_{n+1}\rho=b\rho.$$

Therefore $(a,b)\in\rho$. Hence $\rho^l\subseteq\rho$, i.e., ρ^l is the least left zero congruence. Similar argument shows that ρ^r is the least right zero congruence on S.

Clearly, $\rho^{m'}$ is a congruence on S and $(aba,a)\in\rho^{m'}$ for all a,b in S. It follows that $\rho^{m'}$ is a rectangular band congruence.

In order to show that $\rho^m=\rho^{m'}$, let $(a,b)\in\rho^{m'}$. Then there exist x_i,u_s,v_s,y_j,r_t,k_t in S, where $i=1,2,\cdots,n, s=1,2,\cdots,n+1, j=1,2,\cdots,m, t=1,2,\cdots m+1$, such that

$$au_1=x_1v_1, x_1u_2=x_2v_2,\cdots,x_nu_{n+1}=bv_{n+1}, \tag{1}$$

$$r_1a=k_1y_1, r_2y_1=k_2y_2,\cdots,r_{m+1}y_m=k_{m+1}b. \tag{2}$$

Without loss of generality, we assume that $n\geq m$. By (1) and (2) we have

$$au_1r_1a=x_1v_1k_1y_1, x_1u_2r_2y_1=x_2v_2k_2y_2,\cdots,x_mu_{m+1}r_{m+1}y_m=x_{m+1}v_{m+1}k_{m+1}b,$$
$$x_{m+1}u_{m+2}r_{m+1}y_m=x_{m+1}v_{m+2}k_{m+1}b, x_nu_{n+1}r_{m+1}y_m=bv_{n+1}k_{m+1}b.$$

It implies that $(a,b)\in\rho^m$. Hence $\rho^{m'}\subseteq\rho^m$. Conversely for any $(a,b)\in\rho^m$, there exist x_i,y_i,u_j,v_j in S, where $i=1,2,\cdots,n, j=1,2,\cdots,n+1$, such that

$$au_1a=x_1v_1y_1, x_1u_2y_1=x_2v_2y_2,\cdots,x_nu_{n+1}y_n=bv_{n+1}b. \tag{3}$$

Thus

$$aS\cap x_1S\neq\emptyset, x_1S\cap x_2S\neq\emptyset,\cdots,x_nS\cap bS\neq\emptyset$$

and

$$Sa\cap Sy_1\neq\emptyset, Sy_1\cap Sy_2\neq\emptyset,\cdots,Sy_n\cap Sb\neq\emptyset.$$

It implies that $(a,b)\in\rho^{m'}$, and so $\rho^m=\rho^{m'}$.

Let ρ be a rectangular band congruence on S. For any a,b in S if $(a,b)\in\rho^m$, then by (3) we obtain that

$$a\rho=au_1a\rho=x_1v_1y_1\rho=x_1u_2y_1\rho=\cdots=x_nu_{n+1}y_n\rho=bv_{n+1}b\rho=b\rho, \text{ i.e., } (a,b)\in\rho.$$

Hence ρ^m is the least rectangular band congruence on S. □

Definition 2.2 *Let S be a semigroup and S_r a right ideal of S. If $S\backslash S_r$ is an ideal of S and there is not a non-empty proper subset I of S_r such that I and $S_r\backslash I$ are right ideals of S, then S_r is called a maximal non-decomposable right ideal of S. Symmetrically, we can define maximal non-decomposable left ideal of S.*

Lemma 2.3 *Let S be a semigroup. Then every ρ^l-class (ρ^r-class) in S is a maximal* *non-decomposable right (left) ideal of S.*

Proof we assume that $a \in S$ and I is a non-empty proper subset of $a\rho^l$. Clearly, $a\rho^l$ is a right ideal of S. For any $x \in I$ and $y \in a\rho^l \backslash I$, there exist $x_0, x_1, \cdots, x_n \in a\rho^l$, where $x = x_0$ and $y = x_n$, such that

$$xS \cap x_1 S \neq \emptyset, x_1 S \cap x_2 S \neq \emptyset, \cdots, x_n S \cap yS \neq \emptyset.$$

Let $x_i (1 \leq i \leq n)$ be the first element which does not belong to I. Since $x_{i-1} S \cap x_i S \neq \emptyset, IS \cap (a\rho^l \backslash I)S \neq \emptyset$ and so I or $a\rho^l \backslash I$ is not a right ideal of S. From the definition of ρ^l, for any $x \in S, xS \cap (a\rho^l) \neq \emptyset$ if and only if $x \in a\rho^l$. Thus for any $x \in S \backslash a\rho^l, xS \cap (a\rho^l) = \emptyset$. It shows that $(S \backslash a\rho^l)S \subseteq S \backslash a\rho^l$ and so $S \backslash a\rho^l$ is an ideal of S. We conclude that $a\rho^l$ is a maximal non-decomposable right ideal of S. □

Theorem 2.4 *Let S be a semigroup. Then ρ is a left (right) zero congruence if and only if each ρ-class is a union of maximal non-decomposable right (left) ideal of S.*

Proof A congruence ρ on S is a left zero congruence if and only if $\rho^l \subseteq \rho$, if and only if $a\rho = \cup_{x \in a\rho} x\rho^l$, i.e., each ρ-class is a union of maximal non-decomposable right ideals of S by Lemme 2.3. □

Definition 2.5 *Let Q be a subset of semigroup S. If Q is the meet of a maximal non-decomposable left ideal with a maximal non-decomposable right ideal, then Q is is called a M-quasi-ideal of S.*

Lemma 2.6 *Let S be a semigroup. Then ρ is a rectangular band congruence on S if and only if ρ can be expressed as a meet of a left zero congruence with a right zero congruence.*

Proof Sufficiency. Clear.

Necessity. Let ρ be a rectangular band congruence on S. Thus there exist a left zero semigroup S_L and a right zero semigroup S_R such that $S/\rho \simeq S_L \times S_R$(see [2]). Without loss of generality, suppose $S/\rho = S_L \times S_R$. We define σ_L and σ_R as follows:

For all $a, b \in S, a\sigma_L b$ if and only if there exist $(\lambda, \mu_1), (\lambda, \mu_2) \in S/\rho$ such that

$$a\rho = (\lambda, \mu_1), b\rho = (\lambda, \mu_2).$$

For all $a, b \in S, a\sigma_R b$ if and only if there exist $(\lambda_1, \mu), (\lambda_2, \mu) \in S/\rho$ such that

$$a\rho = (\lambda_1, \mu), b\rho = (\lambda_2, \mu).$$

It is easy to check that σ_L and σ_R are congruences on S. For any $a, b \in S$, we assume $a\rho = (\lambda_1, \mu_1)$ and $b\rho = (\lambda_2, \mu_2)$. Then $ab\rho = (\lambda_1, \mu_2)$. It implies that $(ab, a) \in \sigma_L$ and $(ab, b) \in \sigma_R$, i.e., σ_L and σ_R are a left zero congruence and a right zero congruence on S respectively. Since $(a, b) \in \sigma_L \cap \sigma_R$ if and only if $a\rho = b\rho$, i.e., $(a, b) \in \rho$. Therefore $\rho = \sigma_L \cap \sigma_R$. □

Theorem 2.7 *Let S be a semigroup. Then ρ is a rectangular band congruence on S if and only if each ρ-class is a union of some M-quasi-ideals of S.*

Proof Necessity. Let ρ be a rectangular band congruence on S. By Lemma 2.6 there exist a left zero congruence σ_L and a right zero congruence σ_R such that $\rho = \sigma_L \cap \sigma_R$. By Theorem 2.4 each σ_L-class is a union of maximal non-decomposable right ideals of S and each σ_R-class is a union of maximal non-decomposable right ideals of S. Then for $a \in S$

$$a\rho = a(\sigma_L \cap \sigma_R) = a\sigma_L \cap a\sigma_R = (\cup_{x\in a\sigma_L} x\rho^l) \cap (\cup_{t\in b\sigma_R} t\rho^r) = \cup_{x\in\sigma_L, t\in\sigma_R}(x\rho^l \cap t\rho^r).$$

Hence each ρ-class is a union of some M-quasi-ideals of S.

Sufficency. If each ρ-class is a union of some M-quasi-ideals of S for a congruence ρ on S, then $a\rho = a\rho b\rho a\rho$ is derived by $a, aba \in a\rho^l \cap \rho^r$ for all $a, b \in S$. Hence ρ is a rectangular band congruence. □

Corollary 2.8 *Let S be a semigroup. Then ρ is the least rectangular band congruence on S if and only if each ρ-class is a M-quasi-ideals of S.*

By Corollary 2.8, a M-quasi-ideal A in a semigroup S can be denoted by $a\rho^m$, where $a \in A$.

3. Extending Rectangular band congruence

Let S be a semigroup and T a subsemigroup of S. For a congruence ρ on S, ρ restricted to T is denoted by $\rho\, |_T = \rho \cap (T \times T)$.

Definition 3.1 *Let S be a semigroup, T a subsemigroup of S and σ_T a congruence on T. If there exists a congruence ρ on S such that $\rho\, |_T = \sigma_T$, then ρ is called a congruence extension of σ_T from T to S.*

Lemma 3.2 *Let S be a semigroup, T a subsemigroup of S. If the least rectangular band congruence τ_T on T can be extended to a rectangular band congruence on S, then ρ^m is a congruence extension of τ_T from T to S.*

Proof Suppose that τ_T on T can be extended to a rectangular band congruence σ on S. Since τ_T is the least rectangular band congruence, $\rho^m\, |_T \supseteq \tau_T$. Moreover, $\sigma \supseteq \rho^m$ implies that $\tau_T = \sigma\, |_T \supseteq \rho^m\, |_T$. Thus $\rho^m\, |_T = \tau_T$. □

Lemma 3.3 *Let T be a subsemigroup of a semigroup S. Then the least rectangular band congruence on T can be extended to a rectangular band congruence on S if and only if, for any M-quasi-ideal $a\rho^m$ such that $a\rho^m \cap T \neq \emptyset, a\rho^m \cap T$ is a M-quasi-ideal on T.*

Proof Necessity. Suppose that the least rectangular band congruence on T can be extended to a rectangular band congruence on S. By lemma 3.2, ρ^m is a congruence extension of τ_T. For any M-quasi-ideal $a\rho^m$ of S, if $a\rho^m \cap T \neq \emptyset$, then $a\rho^m \cap T$ is a M-quasi-ideal on T by $\rho^m\, |_T = \tau_T$.

Sufficiency. Let τ_T be the least rectangular band congruence on T. For any a in $T, a\tau_T$ is a M-quasi-ideal of T by Corollary 2.8. Since for any a in T such that

$$a\rho^m \cap T \neq \emptyset,$$

$a\rho^m \cap T$ is a M-quasi-ideal on T, i.e., $a\rho^m \cap a\tau_T = a\tau_T$. Thus $\rho^m\, |_T = \tau_T$. □

Lemma 3.4 *Let T be a subsemigroup of a semigroup S. Define*

$$K_s = \{a\rho^m \mid a \in S, a\rho^m \cap T \neq \emptyset\}.$$

Then $T_k = \cup_{a\rho^m \in K_s} a\rho^m$ is a subsemigroup of S.

Proof For all a, b in T_k, there exist u, v in T such that $u \in a\rho^m$ and $v \in b\rho^m$. Since $ab\rho^m uv, ab\rho^m \cap T \neq \emptyset$. Hence $ab \in T_k$.

Lemma3.5[1] *Let S be a rectangular band . Then each congruence on any subsemigroup of S can be extended to a congruence on S.*

Lemma 3.6[2] *Let ρ and σ be congruences on a semigroup S such that $\rho \subseteq \sigma$. Then*

$$\sigma/\rho = \{(x\rho, y\rho) \in S/\rho \times S/\rho \mid (x, y) \in \sigma\}$$

is a congruence on S/ρ, and $(S/\rho)/\sigma/\rho \cong S/\sigma$.

Lemma 3.7 *Let S be a semigroup, τ a congruence on S/ρ^m. Then for any a, b in S, the relation ρ:*

for any a, b in $S, a\rho b$ if and only if $(a\rho^m, b\rho^m) \in \tau$

is a congruence on S.

Proof Clear.

Theorem 3.8 *Let T be a subsemigroup of a semigroup S. Then every rectangular band congruence on T can be extended to a rectangular band congruence on S if and only if the least rectangular band congruence on T can be extended to a rectangular band congruence on S.*

Proof Necessity. Clear.

Sufficiecy. Suppose the least rectangular band congruence τ_T on T can be extended to a rectangular band congruence on S. By Lemma 3.2, ρ^m is a congruence extension of τ_T. Since $T/\tau_T = T/(\rho^m |_T)$, we can consider T/τ_T as a subsemigroup of S/ρ^m. By Lemma 3.5 every congruence on T/τ_T can be extended to a congruence on S/ρ^m. Since $\sigma_T \supseteq \tau_T$ for any rectangular band congruence σ_T on T , $T/\sigma_T \cong (T/\tau_T)/(\sigma_T/\tau_T)$ by Lemma 3.6. Thus there exists a congruence σ_S/ρ^m on S/ρ^m which is a congruence extention of σ_T/τ_T from T/τ_T to S/ρ^m. Define a relation ρ on S: for any a, b in $S, a\rho b$ if and only if

$$(a\rho^m, b\rho^m) \in \sigma_S/\rho^m. \tag{4}$$

By Lemma 3.7, ρ is a congruence on S. By the definition (4), for all a, b in $T, a\rho b$ if and only if $(a\tau_T, b\tau_T) \in \sigma_T/\tau_T$ if and only if $(a, b) \in \sigma_T$. It is shown that ρ is a congruence extention of σ_T from T to S. Since $a\rho^m b\rho^m a\rho^m = a\rho^m$ for all a, b in $S, (a\rho^m b\rho^m a\rho^m, a\rho^m) \in \sigma_S/\rho^m$ and so $(aba, a) \in \rho$. Hence ρ is a rectangular band congruence. □

From Lemma 3.3 and Theorem 3.8, we obtain

Corollary 3.9 *Let T be a subsemigroup of a semigroup S. Then any rectangular band* *congruence on T can be extended to a rectangular band congruence on S if and only if for any M-quasi-ideal $a\rho^m$ of S such that $a\rho^m \cap T \neq \emptyset, a\rho^m \cap T$ is a M-quasi-ideal of T.* □

Definition 3.10 *Let T be a subsemigroup of a semigroup S. If $a\rho^m \cap T \neq \emptyset$ for any M-quasi-ideal $a\rho^m$ of S, then T is called a M-subsemigroup of S. If T is a M-subsemigroup of S and $a\rho^m \cap T$ is a M-quasi-ideal of T for any M-quasi-ideal $a\rho^m$ of S, then T is called a strong M-subsemigroup of S.*

If S is a regular semigroup, then the subsemigroup $\langle E(S)\rangle$ generated by all idempotents of S is a strong M-subsemigroup. In special, the subsemigroup $E(S)$ of all idempotents of a orthodox semigroup S is a strong M-subsemigroup of S.

Lamma 3.11 *Let S be a semigroup, and T a M-subsemigroup of S. If rectangular band congruences ρ_1, ρ_2 on S satisfy $\rho_1 |_T = \rho_2 |_T$, then $\rho_1 = \rho_2$.*

Proof Let $a\rho_1 b$ for a, b in S. By Corollary 2.8 and T being a M-subsemigroup, each ρ^m-class in S contains the elements of T. Thus there exist a_t, b_t in T such that $a\rho^m a_t, b\rho^m b_t$. Since $\rho^m \subseteq \rho_1, \rho^m \subseteq \rho_2$, we know that $a\rho_1 a_t, b\rho_1 b_t, a\rho_2 a_t$ and $b\rho_2 b_t$ hold. From $a_t\rho_1 a\rho_1 b\rho_1 b_t$ and $\rho_1 |_T = \rho_2 |_T$, we have $a\rho_2 a_t\rho_2 b_t\rho_2 b$. Similarly , we have $a\rho_1 b$ if $a\rho_2 b$. □

Theorem 3.12 *Let S be a semigroup, and T a subsemigroup of S. Then every rectangular band congruence on T can be extended uniquely to a rectangular band congruence on S if and only if T is a strong M-subsemigroup of S.*

Proof Sufficiency. If T is a strong M-subsemigroup of S, then $a\rho^m \cap T \neq \emptyset$ and $a\rho^m \cap T$ is a M-quasi-ideal of T for any a in S. By Lemma 3.3, the least rectangular band congruence on T can be extended to a rectangular band congruence on S. Further, by Theorem 3.8 every rectangular band congruence on T can be extended to a rectangular band congruence on S. From Lemma 3.11, we know that every rectangular band congruence on T can be extended uniquely to a rectangular band congruence on S.

Necessity. Suppose that each rectangular band congruence on T can be extended uniquely to a rectangular band congruence on S. Then the least rectangular band congruence τ_T on T can be extended uniquely to ρ^m and so for any a in $S, a\rho^m \cap T$ is a M-quasi-ideal of T if $a\rho^m \cap T \neq \emptyset$. For any $a\rho^m$ in $K_S, a\rho^m \cap T$ is a M-quasi-ideal of T. To show that T is a strong M-subsemigroup of S, it is enough to prove $T_k = S$, i.e., $(S/\rho^m)\backslash T_k^M = \emptyset$, where $T_k^M = \{a\rho^m \mid a \in T\}$. Suppose that T is not a strong M-subsemigroup of S . Hence $S\backslash T_k \neq \emptyset$ and so ,$(S/\rho^m)\backslash T_k^M \neq \emptyset$. Without loss of generality, assume $S/\rho^m = I \times \Lambda$ (see [2]), where I is a left zero semigroup and Λ is a right zero semigroup. Each subsemigroup in $I \times \Lambda$ can be described as $I_1 \times \Lambda_1$, where $I_1 \subseteq I$ and $\Lambda_1 \subseteq \Lambda$. Let $T_k/(\rho^m |_{T_k}) = I_1 \times \Lambda_1$. Thus $I_1 \subset I$ or $\Lambda_1 \subset \Lambda$. It is clear that $I_1 \times \Lambda_1 \cong T/\tau_T$. Then $I_1 \times (\Lambda\backslash\Lambda_1), (I\backslash I_1) \times \Lambda_1$ and $(I\backslash I_1) \times (\Lambda\backslash\Lambda_1)$ are subsemigroups of S/ρ^m. We denote the universal relation on a semigroup S by Ω_S. Notice that the relation

$$\eta = \Omega_{I_1 \times \Lambda_1} \cup \Omega_{(I\backslash I_1)\times\Lambda_1} \cup \Omega_{I_1 \times (\Lambda\backslash\Lambda_1)} \cup I_{(I\backslash I_1)\times(\Lambda\backslash\Lambda_1)}$$

is a congruence on S/ρ^m and is not universal relation, where $I_{(I\backslash I_1)\times(\Lambda\backslash\Lambda_1)}$ is the identical relation on $(I\backslash I_1) \times (\Lambda\backslash\Lambda_1)$.

Define a relation ρ on S as follows:

for any a, b in $S, a\rho b$ if and only if $(a\rho^m, b\rho^m) \in \eta$.

By Lemma 3.7, we know that ρ is a congruence on S. Since for any a, b in $S, a\rho b\rho a\rho = a\rho$ and η is a congruence on S/ρ^m, we obtain that $(aba\rho^m, a\rho^m) \in \eta$, and so $(aba, a) \in \rho$. It implies that ρ is a rectangular band congruence. Since η is not the universal relation, there exist a, b in S such that $(a\rho^m, b\rho^m) \notin \eta$. Hence $(a, b) \notin \rho$ and so ρ is not the universal relation on S. We have known that $\eta |_{I_1 \times \Lambda_1}$ is the universal relation on $T/(\rho^m |_T) = I_1 \times \Lambda_1 (\cong T/\tau_T)$. Then ,for any a, b in $T, (a\rho^m, b\rho^m) \in \eta$, i.e., $(a, b) \in \rho$. It implies that $\rho |_T$ is the universal relation on T. We have known that the universal relation Ω_T on T can be extended to the universal relation on S. But ρ is not a universal relation on S. Therefore it is not unique that the universal relation Ω_T on T is extended to a rectangular band congruence on S. This is a contradiction with the hypothesis. This means $S/\rho^m = T_K^M$. Thus T is a strong M-subsemigroup of S. □

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半群的子半群上的同余扩张

李 勇 华[1]、张 谋 成[2]

1. 中山大学数学系，广州 510275;

1. 华南师范大学数学系，广州 510631;

2. 华南师范大学数学系，广州 510631

摘 要: 设 ρ 是半群 S 上的一个同余. 如果 S/ρ 是矩形带，则称 ρ 是矩形带同余. 本文刻画了半群上的最小矩形带同余. 设 T 是半群 S 的子半群. 本文给出了 T 上每个矩形带同余能扩张成 S 上矩形带同余的充分必要条件.