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Infinitesimally Stability of C^∞ Map-Germs under a Subgroup of $\mathcal{A}$*

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Abstract: In [1], the universal unfolding of C^∞ map germs under a subgroup of the group $\mathcal{A}$,which is well known group defined by J.Mather [2], was discussed. In this paper, Some conditions are given to characterize the infinitesimally stability of unfoldings.

Key words: map germ; universal unfolding; infinitesimally stability.

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1. Definition notations

Let $E_{n,p}$ be the space of germs at $0 \in \mathbf{R}^n$ of C^∞ mappings from $\mathbf{R}^n$ to $\mathbf{R}^p$. For f in $E_{n,p}$ we will use the notation $f : (\mathbf{R}^n, 0) \to \mathbf{R}^p$. If $f(0) = 0$, we represent f by the notation $f : (\mathbf{R}^n, 0) \to (\mathbf{R}^p, 0)$. The space of such germs will be denoted by $E^0_{n,p}$.

Definition 1 *Two germs $f, g \in E^0_{n,p}$ are isomorphic, if the following diagram commute:*

$$\begin{array}{ccccc} (\mathbf{R}^n,0) & \xrightarrow{f} & (\mathbf{R}^n \times \mathbf{R}^{p-r},0) & \xrightarrow{\pi} & (\mathbf{R}^r,0) \\ \Phi\downarrow & & \Psi\downarrow & & \psi\downarrow \\ (\mathbf{R}^n,0) & \xrightarrow{g} & (\mathbf{R}^n \times \mathbf{R}^{p-r},0) & \xrightarrow{\pi} & (\mathbf{R}^r,0) \end{array}$$

where Φ, Ψ, ψ are germs of diffeomorphisms, π is the projection onto the first factor. Obviously the space of all (Φ, Ψ) forms a subgroup of $\mathcal{A}$.

Definition 2 *Let $f_0 \in E^0_{n,p}$ be a map germ. A q-parameter unfolding of f is a C^∞ map germ*

$$F : (\mathbf{R}^q \times \mathbf{R}^n, 0) \longrightarrow (\mathbf{R}^q \times \mathbf{R}^p, 0)$$

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such that

1) $F(u,x) = (u, f(u,x)), u \in \mathbf{R}^q, x \in \mathbf{R}^n, f(u,x) \in \mathbf{R}^p$,
2) $f_0(x) = f(0,x)$.

Definition 3 *Two q-parameter unfolding F and G of f are called isomorphic if the following diagram commute:*

$$\begin{array}{ccccccccc} (\mathbf{R}^q,0) & \xleftarrow{\pi} & (\mathbf{R}^q\times\mathbf{R}^n,0) & \xrightarrow{F} & (\mathbf{R}^q\times\mathbf{R}^r\times\mathbf{R}^{p-r},0) & \xrightarrow{\pi} & (\mathbf{R}^q\times\mathbf{R}^r,0) & \xrightarrow{\pi} & (\mathbf{R}^q,0) \\ Id\downarrow & & \Phi\downarrow & & \Psi\downarrow & & \psi\downarrow & & \downarrow Id \\ (\mathbf{R}^q,0) & \xleftarrow{\pi} & (\mathbf{R}^q\times\mathbf{R}^n,0) & \xrightarrow{G} & (\mathbf{R}^q\times\mathbf{R}^r\times\mathbf{R}^{p-r},0) & \xrightarrow{\pi} & (\mathbf{R}^q\times\mathbf{R}^r,0) & \xrightarrow{\pi} & (\mathbf{R}^q,0) \end{array}$$

where Id is the identity map germ, Φ, Ψ, ψ are map germs of diffeomorphisms. All π are obvious canonical projections on to the first (or the first two) factor.

We also call Φ (or Ψ) is a q-parameter unfolding of the identity mappings of $\mathbf{R}^n$ (or $\mathbf{R}^p$).

Definition 4 *An unfolding is called trivial if it is isomorphic to the constant unfolding:*

$$(u,x) \longrightarrow (u, f(x)).$$

Definition 5 *Let F be a q-parameter unfolding of f, and let*

$$\begin{aligned} h:\ & (\mathbf{R}^k,0) \longrightarrow (\mathbf{R}^q,0) \\ & \qquad v \longmapsto u = h(v) \end{aligned}$$

be a C^∞ map germ. Define $G = h^*F$, a k-parameter unfolding of f, by

$$\begin{aligned} G:\ & (\mathbf{R}^k\times\mathbf{R}^n,0) \longrightarrow (\mathbf{R}^k\times\mathbf{R}^p,0) \\ & (v,x) \longrightarrow (v, f(h(v),x)). \end{aligned}$$

The unfolding G is called the pull-back of F by h.

Definition 6 *Two unfolding F and G of f are called <u>equivalent</u> if G is isomorphic to h^*F, where h is a germ of diffeomorphism between the parameter spaces of F and G.*

An unfolding F of f is <u>universal</u> if every unfolding of f is isomorphic to h^*F for some mapping h.

Definition 7 *Let $f \in E_{n,p}$ be a map germ. The vector space $Tf \subset E_{n,p}$ defined by*

$$Tf = \{Df\cdot X + Y\circ f \mid X \in X_n,\ Y \in Y_p\}$$

will be called the <u>tangent space at f</u>, where X_n is the space of germs of vector fields at $0 \in \mathbf{R}^n$, Y_n is the subspace of germs of vector fields at $0 \in \mathbf{R}^p$, its elements have the form: $\sum_{i=1}^r V_i(y_1,\dots,y_r)\frac{\partial}{\partial y_i} + \sum_{i=r+1}^p V_i(y)\frac{\partial}{\partial y_i}$.

Now let $f : (\mathbf{R}^n,0) \longrightarrow (\mathbf{R}^p,0)$, this germ defines a homorphism:

$$\begin{aligned} f^*:\ & E_p \longrightarrow E_n \\ & \lambda \longmapsto f^*\lambda = \lambda\circ f, \end{aligned}$$

from the ring of function germs in the target ($\mathbf{R}^p$) to the ring of germs in the source ($\mathbf{R}^n$). This allows us to consider every E_n-module as an E_p-module via the homomorphism f^*.

Let $e_1, \ldots, e_p$ be the canonical basis of the vector space $\mathbf{R}^p$. These vectors may be considered as elements of $E_{n,p}$, and they define a system of generators of the E_n-module $E_{n,p} = (E_n)^p$,

$$E_{n,p} = E_n\{e_1, \ldots, e_p\}.$$

Similarly, we have

$$E^\circ_{n,p} = \mu_n\{e_1, \ldots, e_p\},$$

where μ_n is the maximal ideal of ring E_n.

If $X = (X_1, \ldots, X_n) \in X_n$ is a germ of vector field at $0 \in \mathbf{R}^n$ with components $X_1, \ldots, X_n \in E_n$, we have

$$Df \cdot X = \sum_{i=1}^{n} X_i \frac{\partial f}{\partial x_i} \in E_{n,p}$$

(where $\frac{\partial f}{\partial x_i} \in E_{n,p}$ is the i^{th} column of the Jacobian matrix Df).

If $Y = (Y_1, \ldots, Y_p)$ is a germ of vector field at $0 \in \mathbf{R}^p$, with components $Y_1, \ldots, Y_p$, then

$$Y \circ f = \sum_{i=1}^{p} (Y_i \circ f) \cdot e_i \in E_{n,p}.$$

We can write:

$$Tf = E_n \left\{ \frac{\partial f}{\partial x_i}, \ldots, \frac{\partial f}{\partial x_n} \right\} + E_r\{e_1, \ldots, e_r\} + E_p\{e_{r+1}, \ldots, e_p\}.$$

We denote $E_n \left\{ \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n} \right\}$, $E_r\{e_1, \ldots, e_r\} + E_p\{e_{r+1}, \ldots, e_p\}$ by $J(f), \tau(f)$ respectively, $Tf = J(f) + \tau(f)$, this is the algebraic description of the tangent space Tf.

Definition 8 *A germ $f \in E^\circ_{n,p}$ is of finite type if the quotient space $E_{n,p}/Tf$ is a finite generated E_p-module. This means that there exist germs $g_1, \ldots, g_t \in E_{n,p}$, such that:*

$$Tf + E_p\{g_1, \ldots, g_t\} = E_{n,p}.$$

Definition 9 *A germ $f \in E^\circ_{n,p}$ is called infinitesimally stable, if $Tf = E_{n,p}$.*

2. Main results

Proposition 1 *Let $f_0 : (\mathbf{R}^n, 0) \longrightarrow (\mathbf{R}^p, 0)$ be a map germ of finite codimension q, then*

(1) All of the q-parameter universal unfolding of f_0 are equivalent (these are called minimal).

(2) Any universal unfolding of f_0, with $t > q$ parameters, is equivalent to the constant unfoldings (with $t - q$ parameters) of a minimal universal unfolding.

Proof (1) Let F and G be two q-parameter universal unfoldings of f_0, then there is a germ of a C^∞ map

$$h: \ (\mathbf{R}^q,0) \longrightarrow (\mathbf{R}^q,0)$$
$$v \longmapsto u = h(v).$$

Such that G is isomorphic to $H = h^*F$. Since G is universal, so is H. We have $\dot{H}_i = \sum_{j=1}^q \frac{\partial h_j}{\partial v_i} \cdot \dot{F}_j$, $*codim f_0 = q$, $)Tf_0 + \mathbf{R}\{\dot{F}_1, \ldots, \dot{F}_q\} = E_{n,p}$, $Tf_0 + \mathbf{R}\{\dot{H}_1, \ldots, \dot{H}_q\} = E_{n,p}$,) the systems $\{\dot{H}_1, \ldots, \dot{H}_q\}$ and $\{\dot{F}_1, \ldots, \dot{F}_q\}$ are both linearly independent over $\mathbf{R}$, so the matrix $Dh(0)$ is invertible and h is a local diffeomorphism.

(2) Samely, if F is a q-parameter universal unfolding of f_0, $q = codim f_0$, and G is a t-parameter universal unfolding of f_0, $t > q$, then G is isomorphic to h^*F, where $h: (\mathbf{R}^q,0) \longrightarrow (\mathbf{R}^q,0)$ is a submersion, i.e. $Dh(0)$ is surjective. Thus G is equivalent to a constant unfolding of F. □

Theorem 2 *Let $F : (\mathbf{R}^q \times \mathbf{R}^n, 0) \longrightarrow (\mathbf{R}^q \times \mathbf{R}^p, 0)$ be a q-parameter unfolding of $f_0 : (\mathbf{R}^n,0) \longrightarrow (\mathbf{R}^p,0)$, $f_0(x) = f(0,x)$. Then F is infinitesimally stable if and only if*

(1) $$Tf_0 + E_p\{\dot{F}_1, \ldots, \dot{F}_q\} = E_{n,p}.$$

Proof That F is infinitesimally stable, by definition, means

(2) $$TF = (E_{u,x})^{q+p},$$

where $E_{u,x}$ (resp. $E_{u,y}$) stand for the ring of function germs at the origin of $\mathbf{R}^q \times \mathbf{R}^n$ (resp. $\mathbf{R}^q \times \mathbf{R}^p$). (2) means that

(3) $$DF \cdot \overline{X} + Y \circ F = \overline{Z},$$

where $\overline{X} = \binom{\xi}{X}$, $\overline{Y} = \binom{\mu}{Y}$, $\overline{Z} = \binom{\zeta}{Z}$, $\xi\,(\mu)$ is the component of $\overline{X}\,(\overline{Y})$ in the parameter space $\mathbf{R}^q$. $X\,(Y)$ is the component of $\overline{X}\,(\overline{Y})$ in $\mathbf{R}^n(\mathbf{R}^p)$. $\overline{X}\,(\overline{Y})$ is a vector field in $\mathbf{R}^q \times \mathbf{R}^n(\mathbf{R}^q \times \mathbf{R}^p)$, and Y has the form:

$$Y = \sum_{i=1}^{r} Y_i(u, y_1, \ldots, y_r)\frac{\partial}{\partial y_i} + \sum_{i=r+1}^{p} y_i(u,y)\frac{\partial}{\partial y_i}.$$

In a similar way, we write any element $\overline{Z}$ of $(E_{u,x})^{q+p}$ as $\overline{Z} = \binom{\zeta}{Z}$

(4) $$\left(\begin{array}{c|c} I & 0 \\ \hline \frac{\partial f}{\partial u_1}, \ldots, \frac{\partial f}{\partial u_q} & \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n} \end{array}\right) \begin{pmatrix} \xi \\ X \end{pmatrix} + \begin{pmatrix} \mu \circ F \\ Y \circ F \end{pmatrix} = \begin{pmatrix} \zeta \\ Z \end{pmatrix}$$

or

(5) $$\begin{cases} \xi = \zeta - \mu \circ F, \\ -\sum\limits_{i=1}^{q} (\mu_i \circ F) \cdot \frac{\partial f}{\partial u_i} + \sum\limits_{i=1}^{n} X_i \frac{\partial f}{\partial x_i} + Y \circ F = Z - \sum\limits_{i=1}^{q} \zeta_i \circ \frac{\partial f}{\partial u_i}, \end{cases}$$

where $\zeta = (\zeta_1, \dots, \zeta_q) \in (E_{u,y})^q$, $\mu = (\mu_1, \dots, \mu_q) \in (E_{u,y})^q$,

$$X = (X_1, \dots, X_n) \in (E_{u,x})^n.$$

It is clear that (3) has a solution $(\overline{X}, \overline{Y})$ for any $\overline{Z}$ if and only if the eqation:

$$\sum_{i=1}^{q}(\mu_i \circ F) \cdot \frac{\partial f}{\partial u_i} + \sum_{i=1}^{n} X_i \cdot \frac{\partial f}{\partial x_i} + Y \circ F = Z \tag{6}$$

has a solution (X, Y, μ) for any Z in $(E_{u,x})^p$.

Now note that the set

$$\left\{\sum_{i=1}^{n} X_i \frac{\partial f}{\partial x_i} + Y \circ F \mid X_i \in E_{u,x};\ Y \in (E_{u,y})^p\right\}$$

represents the unfolding $\overline{T}F$ of the tangent space Tf_0 associated with F. Thus, condition (6) means that the quotient

$$M = (E_{u,x})^p / \overline{T}F$$

is an $E_{u,y}$-module of finite type, generated by the images of $\frac{\partial f}{\partial u_1}, \dots, \frac{\partial f}{\partial u_q}$.

Using Theorem X 6.3. [Martinet], this is equivalent to

$$M_0 = M/u \cdot M = (E_x)^p / Tf_0$$

is an E_y-module of finite type, generated by the projections of $\left(\frac{\partial f}{\partial u_1}\right)_0 = \dot{F}_1, \dots, \left(\frac{\partial f}{\partial u_2}\right)_0 = \dot{F}_q$.

But this is exactly condition (1). □

Proposition 3 *Let $f \in E^0_{n,p}$ and $g_1, \dots, g_q \in E_{n,p}$. The following conditions are equivalent:*

(i) $Tf + E_p\{g_1, \dots, g_q\} = E_{n,p}$,

(ii) $J(f) + f^*\mu_r \cdot E_{n,r} + f^*\mu_p \cdot E_{n,p-r} + \mathbf{R}\{e_1, \dots, e_p,\ g_1, \dots, g_q\} = E_n$.

Remark The meaning of the sum $f^*\mu_r \cdot E_{n,r} + f^*\mu_p \cdot E_{n,p-r}$ is easily understood, we don't explain it here.

Proof By definition $Tf = J(f) + E_r\{e_1, \dots, e_r\} + E_p\{e_{r+1}, \dots, e_p\}$. Therefore (i) means that

$$J(f) + E_r\{e_1, \dots, e_r\} + E_p\{e_{r+1}, \dots, e_p,\ g_1, \dots, g_q\} = E_{n,p},$$

that it is equivalent to (ii) follows immediately from the preparation Theorem X 2.3. [Martinet], applied to the finitely generated E_n-module $E_{n,p}/J(f)$. □

Theorem 4 *Let F be a q-parameter unfolding of $f_0 \in E^0_{n,p}$, suppose f_0 has rank0 at $0 \in \mathbf{R}^n$, and $F_i(0) = 0$, then the following conditions are equivalent:*

(a) *F is an infinitesimally stable germ.*

(b) $J(f_0) + f_0^*\mu_r \cdot E_{n,r} + f_0^* \cdot E_{n,p-r} + \mathbf{R}\{\dot{F}_1, \dots, \dot{F}_q\} = E^0_{n,p}$.

(c) $J(f_0)+f_0^*\mu_r\cdot E_{n,r}+f_0^*\mu_p\cdot E_{n,p-r}+\mu_n^{q+2}\cdot E_{n,p}+\mathbf{R}\{\dot F_1,\dots,\dot F_q\}=E^0_{n,p}$.

Proof Let

$$F:\quad (\mathbf{R}^{q+n},0)\longrightarrow(\mathbf{R}^{q+p},0)$$
$$(u,x)\longmapsto(u,f(u,x)),$$

where $u\in\mathbf{R}^q$, $x\in\mathbf{R}^n$, $f(u,x)\in\mathbf{R}^p$.

Because f_0 has $rank0$ at $0\in\mathbf{R}^n$, since we may suppose that its Jacobian can be normalized as

$$DF(0)=\left(\begin{array}{c|c} \overbrace{I}^{q} & \overbrace{0}^{n}\ \}\,q \\ \hline 0 & 0\ \}\,p \end{array}\right)$$

so we have

$$f_0\in\mu_n^2\cdot E_{n,p}\qquad (f_0(x)=f(0,x)),$$
$$\dot F_i\in\mu_n\cdot E_{n,p}=E^0_{n,p}\qquad \left(\dot F_i(x)=\frac{\partial f}{\partial u_i}(0,x)\right).$$

Then the infinitesimal stability of F is equivalent to:

(1) $$Tf_0+E_p\{\dot F_1,\dots,\dot F_q\}=E_{n,p}\quad\text{(by Theorem 2.)}.$$

Using Proposition 3. (1) is equivalent to:

(2) $$J(f_0)+f_0^*\mu_r\cdot E_{n,r}+f_0^*\mu_p\cdot E_{n,p-r}+\mathbf{R}\{e_1,\dots,e_p,\ \dot F_1,\dots,\dot F_q\}=E_{n,p}.$$

Because

$$J(f_0)\subset\mu_n\cdot E_{n.p}=E^0_{n,p},$$
$$\mathbf{R}\{\dot F_1,\dots,\dot F_q\}\subset E^0_{n,p},$$
$$f_0^*\mu_r\cdot E_{n,r}+f_0^*\mu_p\cdot E_{n,p-r}\subset E^0_{n,p}\quad(*f_0^*\mu_p\subset\mu_n^2),$$

since (2) is equivalent to:

$$J(f_0)+f_0^*\mu_r\cdot E_{n,r}+f_0^*\mu_p\cdot E_{n,p-r}+\mathbf{R}\{\dot F_1,\dots,\dot F_q\}=E^0_{n,p}.$$

(3) $$(\text{since } E_{n,p}=E^0_{n,p}\oplus\mathbf{R}\{e_1,\dots,e_p\}).$$

Namely, (a) is equivalent to (b).

(b) $\Rightarrow$ (c) is obvious.

(c) $\Rightarrow$ (b) is a consequence of Nakayama's Lemma. Indeed, consider $M=J(f_0)+f_0^*\mu_r\cdot E_{n,r}+f_0^*\mu_p\cdot E_{n,p-r}$, it is an E_n-submodule of $E^0_{n,p}$. Consider the sequence of inclusions of E_n-modules:

$$E^0_{n,p}\underset{c_1}{\supset}\mu_n\cdot E^0_{n,p}+M\underset{c_2}{\supset}\cdots\underset{c_{q+1}}{\supset}\mu_n^{q+1}\cdot E^0_{n,p}+M.$$

Denote by c_i the codimension of $\mu_n^{i+1} \cdot E_{n,p}^0 + M$ in $\mu_n^i \cdot E_{n,p}^0 + M$ $(1 \leq i \leq q+1)$, then $c_1 = 0$ implies, by Nakayama's Lemma, that $M \supset \mu_i^i \cdot E_{n,p}^0$, so that $c_{i+1} = c_{i+2} = \cdots = 0$ and $codim(\mu_n^i \cdot E_{n,p}^0 + M) = c_1 + \cdots + c_i$ condition (c) implies that $\mu_n^{q+1} \cdot E_{n,p}^0 + M$ has *codimension* $\leq q$, therefore $c_1 + \ldots + c_{q+1} \leq q$, so one must have $c_{q+1} = 0$, thus

$$M \supset \mu_n^q \cdot E_{n,p}^0$$

and the proposition follows. □

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光滑映射芽关于群 $\mathcal{A}$ 的一个子群的无穷小稳定性

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摘要: 在 [1] 中， C^∞ 映射芽在 Mather 定义的群 $\mathcal{A}$[2] 中的一个子群下的万有开折得到了讨论，本文则刻画了开折的无穷小稳定性.