Journal of Mathematical Research & Exposition
Vol.19, No.3, 549–553, August 1999

Wiener-Hopf Operators on Discrete Abelian Groups *

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Abstract: The groupoid approach to study Wiener-Hopf operators on discrete abelian groups is realised.

Key words: groupoid; C^*-algebra; discrete abelian group.

Classification: AMS(1991) 47B35/CLC O177.1

Document code: A **Article ID**: 1000-341X(1999)03-0549-05

1. Introduction

The classical theory of Toeplitz operators and their associated C^*-algebras is an elegant and important area of modern mathematics. For this reason, many authors have sought to extend this theory to more general setting. The guises in the literature lie mainly in two respects: One is the Toeplitz operators on the Bergman space in $\mathbf{C}^N$, the other is the Toeplitz operators on the Hardy space in $\mathbf{C}^N$ or in general discrete groups. There is a great difference between these two kinds of spaces, so do the properties of the corresponding Toeplitz operators. It is the custom when we talk about the Toeplitz operators, we must make it clear which space is undertaking. On the other hand, as shown in section 2, on some domains in $\mathbf{C}^N$, such as Reinhardt domains, both these two Toeplitz C^*- algebras can be viewed as the C^*- algebras generated by some weighted shifts. In this paper, we use the groupoid approach to study unitedly these two C^*- algebras under the category of discrete abelian groups.

2. Wiener-Hopf operators on discrete abelian groups

Let G be a discrete abelian group and G_+ a subset of G, (G,G_+) is said to be a quasi-partial ordered group if $0\in G_+$, $G_+ + G_+ \subseteq G_+$ and $G = G_+ - G_+$. If furthermore $G_+^\circ = G_+\cap(-G_+) = \{0\}$, then (G,G_+) is known as the usual partially ordered group, see [4] for example. Let $\{e_g|g\in G\}$ be the usual orthonormal basis for $\ell^2(G)$. In this paper we always assume that G is countable, so it is second countable.

*Received date: 1996-11-21
Foundation item: Youth Science Foundation of Shanghai Higher Education (98QN75)
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Let $w_+ : G_+ \times G_+ \to (0,1]$ be a weighted function satisfying

$$w_+(g_1+g_2, g_3) = w_+(g_1, g_2+g_3)w_+(g_2, g_3) \tag{1}$$

for any $g_i \in G_+$, i=1, 2, 3.

For any $g \in G_+$, define w_g and T_{-g} in $\mathbf{B}(\ell^2(G_+))$ as follows:

$$(w_g\xi)(h) = w_+(g,h)\xi(h),$$

$$(T_{-g}\xi)(h) = \begin{cases} \xi(h-g), & \text{if } h-g \in G_+, \\ 0, & \text{otherwise} \end{cases}$$

for $h \in G_+$ and $\xi \in \ell^2(G_+)$.

Definition *The C^*- algebra generated by $\{T_{-g}, w_g | g \in G_+\}$ is denoted by $W(G)$, which is called the Wiener-Hopf algebra.*

Remark (1) When $w_+ \equiv 1$ on $G_+ \times G_+$, $W(G)$ is just the Toeplitz algebra on the generalized Hardy space, see [4] and [6] for example. If furthermore $G = G_+$, then $W(G)$ is just the group C^*-algebra. (2) When $(G, G_+) = (Z^n, Z^n_+)$ and $w_+(k,l) = \|z^{k+l}\|_{\ell^2(\Omega)}/\|z^l\|_{\ell^2(\Omega)}$ for some bounded Reinhardt domain Ω in $\mathbf{C}^N$, $W(G)$ is just the groupoid C^*-algebra considered in [2].

For $g \in G_+$, let $w(g) \in \ell^\infty(G)$ be the zero-extension of $w_+(g,\cdot)$ i.e., $w(g)(h) = w_+(g,h)$ if $h \in G_+$; $w(g)(h) = 0$ if $h \notin G_+$, define $(\tau_h w(g))(l) = w(g)(h+l)$ for $l \in G$. The C^*-subalgebra of $\ell^\infty(G)$ generated by $\{\tau_h w(g) | g \in G_+, h \in G\}$ is denoted by $\mathcal{A}$ and its maximal ideal space is denoted by Y. Clearly G can be imbeded in Y through evalution (the action here is denoted by α), and let $X = \overline{\alpha(G_+)}^{W^*}$. As shown in [1], [2], $\alpha(G)$ is dense in Y.

Proposition 1 *(1) If $G_+^\circ = \{0\}$, then α is one-to-one.*

(2) $\forall g \in G$, $\alpha(g) \in X$ if and only if $g \in G_+$.

(3) X is both open and compact.

Proof (1) Suppose that x_1 and $x_2 \in G$, such that $\alpha(x_1) = \alpha(x_2)$. Let $g_1 = g_2 = 0$, then by equation (1) we know that $w(0)(h) = 1$ if $h \in G_+$; $w(0)(h) = 0$ otherwise. It follows that $1 = \alpha(x_1)(\tau_{-x_1}w(0)) = \alpha(x_2)(\tau_{-x_1}w(0))$, so $x_2 - x_1 \in G_+$. Similarly $x_1 - x_2 \in G_+$, so $x_1 - x_2 \in G_+ \cap (-G_+) = G_+^0 = \{0\}$.

(2) Let $g \in G$, if $\alpha(g) \in X$, then since $X = \overline{\alpha(G_+)^{W^*}}$, we know that $\alpha(g)(w(0)) = 1$, which implies that $g \in G_+$.

(3) Let $E = \{y \in Y | \widehat{w(0)}(y) > \frac{1}{2}\}$, where $\widehat{w(0)}$ is the Gelfand transformation of $W(0)$, then $X \subseteq E$. On the other hand, if $y \in E$, then since $\alpha(G)$ is dense in Y, there is a net $\{\alpha(g_\lambda)\}_{\lambda \in \wedge}$ such that $\alpha(g_\lambda) \longrightarrow y$ in W^*–topology, especially $\alpha(g_\lambda)(w(0)) \longrightarrow y(w(0)) = \widehat{w(0)}(y) > \frac{1}{2}$, so there is a $\lambda_0 \in \wedge$, such that $g_\lambda \in G_+$ for all $\lambda \geq \lambda_0$, so $y \in X$. Thus we have $X = \{y \in Y | \widehat{w(0)}(y) > \frac{1}{2}\}$, so X is open in Y. The proof above shows that $X = \{y \in Y | \widehat{w(0)}(y) \geq \frac{1}{2}\}$, so X is also closed in Y. Therefore, the characteristic function of X, 1_X is in $C_0(Y)$. It follows that $C_0(X) = (C_0(Y)|_X)$ has an identity and X

must therefore be compact.

Remark When $(G,G_+) = (Z^n, Z^n_+)$ and $w_+(k,l) = \|z^{k+l}\|_{\ell^2(\Omega)}/\|z^l\|_{\ell^2(\Omega)}$ for some bounded Reinhardt domain Ω in $\mathbf{C}^N$, for the sake of proving X's compactness, the condition of jointly bounded below given in [2] is redundant.

Next we prove that $W(G)$ is isomorphic to some groupoid C^*-algebra. We follow the notation and terminology of [5].

Define a map $Y \times G \to Y$, the image of (y,g) is denoted by $y+g$: $(y+g)(a) = y(\tau_g(a))$ for $a \in \mathcal{A}$. It is noticable that by definition we know that, if $\{y_\alpha\}_{\alpha\in\wedge}$ is a net in Y such that $y_\alpha \to y$, then $y_\alpha + g \to y+g$ for any $g \in G$. As in [1] and [2], $Y \times G$ is a transformation groupoid with a natural left Haar system $\{\lambda^y = \delta_y \times \lambda | y \in Y\}$. Let $\mathcal{G} = Y \times G|_X$ the reduction of $Y \times G$ to X, endowed with the natural system of measures, namely $\lambda^x = \delta_x \times \lambda$ where λ is the counting measure on (possibly part of) G and δ_x is the Dirac measure at x in X. Since X is open and G is discrete, by ([3], Proposition 1.3) we know that $\{\lambda^x | x \in X\}$ is actually a left Haar system for $\mathcal{G}$, and since G is abelian, it is amenable which implies $C^*(\mathcal{G}) \cong C^*_{\text{red}}(\mathcal{G})$, and $\text{Ind}\delta_{\alpha(0)}$ induces a representation of $C^*_{\text{red}}(\mathcal{G})$ on $\ell^2(\mathcal{G}_{\alpha(0)})$, where $\delta_{\alpha(0)}$ is the point measure and $\mathcal{G}_{\alpha(0)} = \{(\alpha(t),-t) | t \in G_+\}$ (See [1], Propositions 2.15 and Proposition 2.17). As shown in [2], $\ell^2(\mathcal{G}_{\alpha(0)})$ is isomorphic to $\ell^2(G_+)$ in a natural way, so if we denote the isomorphism by V , then $\pi = V \circ \text{Ind}\,\delta_{\alpha(0)}(\cdot) \circ V^*$ be a representation of $C^*(\mathcal{G})$ on $\ell^2(G_+)$.

For $f \in C_c(\mathcal{G})$ and $\xi \in \ell^2(G_+)$,

$$\begin{aligned}(\pi(f)\xi)(g) &= (\text{Ind}\,\delta_{\alpha(0)}(f)\,V^*\xi)(\alpha(g),-g) = (f * V^*\xi)(\alpha(g),-g) \\ &= \sum_{h\in G_+} f(\alpha(g),h-g)(V^*\xi)(\alpha(h),-h) = \sum_{h\in G_+} f(\alpha(g),h-g)\xi(h).\end{aligned}$$

As shown in [2], we have the following Proposition:

Proposition 2 $\pi(C^*(\mathcal{G})) = W(G)$.

Proof Let $g \in G_+$, write $\widehat{w(g)}$ for the function on $\mathcal{G}$ which is the Gelfand transform of $w(g)$ restricted to X, viewed as the subset $X \times \{0\}$ in $\mathcal{G}$, and which is zero off $X \times \{0\}$. For $g \in G_+$,define $u_+(g) \in C_c(\mathcal{G})$:

$$u_+(g)(x,h) = \begin{cases} 1, & \text{if } h = -g, \\ 0, & \text{otherwise.} \end{cases}$$

For g_1, g_2 and $h \in G_+$, we have

$$u_+(g_1)^* * u_+(g_2) * \widehat{w(h)} * u_+(g_2)^* * u_+(g_1) = \tau_{g_1-g_2}\widehat{w(h)}. \tag{2}$$

Case 1 $(x,g) \in \mathcal{G}$, $x \in X, x+g \in X, x+g_1-g_2 \in X$.

$$\begin{aligned}&(u_+(g_1)^* * u_+(g_2) * \widehat{w(h)} * u_+(g_2)^* * u_+(g_1))(x,g) \\ &= \int u_+(g_1)^*[(x,g)(y,l)][u_+(g_2) * \widehat{w(h)} * u_+(g_2)^* * u_+(g_1)](y+l,-l)\,d\,\lambda^{x+g}(y,l)\end{aligned}$$

$$= \sum_l u_+(g_1)^*(x, g+l)[u_+(g_2) * \widehat{w(h)} * u_+(g_2)^* * u_+(g_1)](x+g+l, -l)$$
$$= \sum_l \overline{u_+(g_1)(x+g+l, -g-l)}[u_+(g_2) * \widehat{w(h)} * u_+(g_2)^* * u_+(g_1)](x+g+l, -l)$$
$$= [u_+(g_2) * \widehat{w(h)} * u_+(g_2)^* * u_+(g_1)](x+g_1, g-g_1)$$
$$= \int u_+(g_2)[(x+g_1, g-g_1)(y,l)][\widehat{w(h)} * u_+(g_2)^* * u_+(g_1)](y+l, -l)\, d\lambda^{x+g}(y,l)$$
$$= \sum_l u_+(g_2)(x+g_1, g-g_1+l)[\widehat{w(h)} * u_+(g_2)^* * u_+(g_1)](x+g+l, -l)$$
$$= [\widehat{w(h)} * u_+(g_2)^* * u_+(g_1)](x+g_1-g_2, g_2-g_1+g)$$
$$= \int \widehat{w(h)}[(x+g_1-g_2, g_2-g_1+g)(y,l)][u_+(g_2)^* * u_+(g_1)](y+l, -l)\, d\lambda^{x+g}(y,l)$$
$$= \sum_l \widehat{w(h)}(x+g_1-g_2, g_2-g_1+g+l)[u_+(g_2)^* * u_+(g_1)](x+g+l, -l)$$
$$= \widehat{w(h)}(x+g_1-g_2) \int u_+(g_2)^*[(x+g_1-g_2, g_2-g_1+g)(y,l)] \times$$
$$u_+(g_1)(y+l, -l) \mathrm{d}\lambda^{x+g}(y,l)$$
$$= \widehat{w(h)}(x+g_1-g_2) \sum_l u_+(g_2)^*(x+g_1-g_2, g_2-g_1+g+l) u_+(g_1)(x+g+l, -l)$$
$$= \widehat{w(h)}(x+g_1-g_2) \sum_l u_+(g_2)(x+g+l, -g_2+g_1-g-l) u_+(g_1)(x+g+l, -l)$$
$$= \widehat{w(h)}(x+g_1-g_2) u_+(g_2)(x+g+g_1, -g_2-g)$$
$$= \begin{cases} 0, & \text{if } g \neq 0, \\ \widehat{w(h)}(x+g_1-g_2), & \text{otherwise.} \end{cases}$$

Case 2 $(x,g) \in \mathcal{G}$, $x \in X, x+g \in X, x+g_1-g_2 \notin X$. By the computation above, we know that in this case, $[u_+(g_1)^* * u_+(g_2) * \widehat{w(h)} * u_+(g_2)^* * u_+(g_1)](x,g) = 0$.

Now for $\widehat{\tau_{g_1-g_2} w(h)}$, if $x \in X, x+g_1-g_2 \in X$, then $\widehat{\tau_{g_1-g_2} w(h)}(x,0) = \widehat{\tau_{g_1-g_2} w(h)}(x) = x(\tau_{g_1-g_2} w(h)) = (x+g_1-g_2) w(h) = \widehat{w(h)}(x+g_1-g_2) = [u_+(g_1)^* * u_+(g_2) * \widehat{w(h)} * u_+(g_2)^* * u_+(g_1)](x,0)$. On the other hand, if $x \in X, x+g_1-g_2 \notin X$, then there is a net $\{\alpha(g_\lambda)\}_{\lambda \in \wedge}$ with $g_\lambda \in G_+$ such that $\alpha(g_\lambda) \to x$, so $\alpha(g_\lambda)+g_1-g_2 \to x+g_1-g_2$. Since X^C is open in Y, there is a $\lambda_0 \in \wedge$, such that $\alpha(g_\lambda)+g_1-g_2 \notin X$ for all $\lambda > \lambda_0$. By Proposition 1 we know that $g_\lambda + g_1 - g_2 \notin G_+$ for all $\lambda > \lambda_0$, therefore

$$\widehat{\tau_{g_1-g_2} w(h)}(x,0) = \widehat{\tau_{g_1-g_2} w(h)}(x) = x(\tau_{g_1-g_2} w(h))$$
$$= \lim_\lambda \alpha(g_\lambda)(\tau_{g_1-g_2} w(h)) = \lim_\lambda w(h)(g_\lambda + g_1 - g_2) = 0.$$

Let $f \in C_c(\mathcal{G})$ with $\mathrm{Supp} f \subseteq X \times \{g\}$ for some $g \in G$, f is defined on $X_g \times \{g\}$, where $X_g = \{x | x \in X, x+g \in X\}$, it is a closed (hence compact) subset of X. Let ϕ be a function defined on $X_g + \{g\}$ by $\phi(x) = f(x-g, g)$, and extend it to X which is still denoted by ϕ. Let $g = g_1 - g_2$, $g_1, g_2 \in G_+$, then it is easy to show that

$$u_+(g_1)^* * u_+(g_2) * \phi = f. \tag{3}$$

By (2) and (3) , we know that $C_c(\mathcal{G})$ is generated by $\{u_+(g), \widehat{w(g)} | g \in G_+\}$. It is easy to show that $\pi(\widehat{w(g)}) = w_g$ and $\pi(u_+(g)) = T_{-g}$, it follows that π carries $C^*(\mathcal{G})$ onto $W(G)$. On the other hand, π is faithful since the smallest closed invariant subset of X containing the support of $\delta_{\alpha(0)}$ is X (see [1], Propositions 2.15 and 2.17, also [2], Theorem 2.5).

Proposition 3 *Suppose $\{g_1, g_2, \ldots, g_n\}$ is a finite subset of $G \backslash G_+$. If for any $g \in G_+$, there is a g_i, $1 \le i \le n$, such that $g + g_i \in G_+$, then $\mathbf{K}(\ell^2(G_+)) \subseteq W(G)$, where $\mathbf{K}(\ell^2(G_+))$ is the ideal of compact operators on $\ell^2(G_+)$.*

Proof By assumption, we know that $\{\alpha(0)\} = \bigcap_{i=1}^n \{x \in X | \tau_{g_i}\widehat{w(0)}(x) < \frac{1}{2}\}$, so $\{\alpha(0)\}$ is open in X, it follows that $1_{\{\alpha(0)\}} \in C(X)$. It is easy to show that $\pi(1_{\{\alpha(0)\}})$ is a rank one projection. Therefore, it remains only to show that $W(G)$ is irreducible on $\ell^2(G_+)$. In fact, if $T \in \mathbf{B}(\ell^2(G_+))$ such that $TS = ST$ for any $S \in W(G)$, then $TT_{-g} = T_{-g}T$ and $T(T_{-g})^* = (T_{-g})^*T$ for any $g \in G_+$. Let $Te_0 = \sum_{h \in G_+} \xi_h^{(0)} e_h$, then easily shows that $T = \xi_0^{(0)} 1$, so $W(G)$ is irreducible on $\ell^2(G_+)$.

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离散交换群上的 Wiener-Hopf 算子

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