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A Note on the Way-Below Relation on LSC$(X, \mathbf{R}^*)$ *

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Abstract: A mistake in Proposition I-1.21.1 of "A Compendium of Continuous Lattices" by G.Giers et al. is pointed out and a revised proposition with a proof is given.

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Let X be a topological space, $\mathbf{R}^*$ the extended real line, i.e. $\mathbf{R}^* = \mathbf{R} \cup \{+\infty\} \cup \{-\infty\}$, where $\mathbf{R}$ is the real set. LSC$(X, \mathbf{R}^*)$ denotes the set of all lower semicontinuous functions from X to $\mathbf{R}^*$. It is easy to prove that LSC$(X, \mathbf{R}^*)$ is a complete lattice with respect to the pointwise order. In the literature "A Compendium of Continuous Lattices" the authors gave some equivalence characters for the way-below relation $\ll$ on LSC$(X, \mathbf{R}^*)$ in order to prove that if X is a compact topological space, then LSC$(X, \mathbf{R}^*)$ is a continuous lattice (see [1], Proposition I-1.21.1). One conclusion in [1], Proposition I-1.21.1 is the following

Conclusion If X is a compact space, then $f \ll g$ holds in LSC$(X, \mathbf{R}^*)$ if and only if there exists a continuous function h from X to $\mathbf{R}^*$ such that $f(x) \leq h(x) \ll g(x)$ for any $x \in X$.

In this paper we point out by a counterexample that the sufficiency of the above conclusion is false and give a revised proposition with a proof.

First at all we note that LSC$(X, \mathbf{R}^*)$ and LSC$(X, \mathbf{I})$ are lattice-isomorphic, where LSC$(X, \mathbf{I})$ denotes the set of all lower semicontinuous functions from X to the unit interval $\mathbf{I} = [0, 1]$, which is a complete lattice with respect the pointwise order. Hence for the convenience we replace $\mathbf{R}^*$ by $\mathbf{I}$ in the sequel.

Counterexample Let $X = [0,1]$ with the ordinary topology. Define $f, g, h : X \to \mathbf{I}$ as follows:

$$f(x) = h(x) = x/2, g(x) = x \text{ for any } x \in X.$$

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Obviously, f, g, h are continuous. This implies $f, g \in \mathrm{LSC}(X, \mathbf{I})$. Since in the complete lattice $\mathbf{I}$, $s \ll t$ if and only if $s < t$ or $s = 0$, so we have $f(x) \leq h(x) \ll g(x)$ for any $x \in X$. Next we prove that $f \ll g$ does not hold.

For any natural number n, define a function $h_n : X \to \mathbf{I}$ as follows: if $i = 1, 2, \cdots, n-1$ and $x \in \left(\frac{i}{n}, \frac{i+1}{n}\right]$, then $h_n(x) = \frac{i}{n}$; if $x \in \left[0, \frac{1}{n}\right]$, then $h_n(x) = 0$. It is easy to see that each h_n is lower semicontinuous and $\{h_n\}_{n \in N}$ is a directed set in $\mathrm{LSC}(X, \mathbf{I})$ and $g = \sup\limits_{n \in N} h_n$. But $f \leq h_n$ does not hold for any $n \in N$. From the definition of way-below relation $\ll$ it follows that $f \ll g$ does not hold.

The correct conclusion should be the following

Proposition *Let X be a compact space, $f, g \in \mathrm{LSC}(X, \mathbf{I})$. Then the following conditions are equivalent:*

(1) $f \ll g$.

(2) there exists a continuous function h from X to $\mathbf{I}$ such that $f(x) \leq h(x) \ll g(x)$ for any $x \in X$ and $g^{-1}(0) \subset \mathrm{Int} f^{-1}(0)$, where Int denotes the interior operator in X.

Proof (2)$\Rightarrow$(1) Let $\{h_j\}_{j \in J}$ be a directed set in $\mathrm{LSC}(X, \mathbf{I})$ and $g \leq \sup\limits_{j \in J} h_j$. Taking $x \in X$, then $h(x) \ll g(x)$. We have the following two cases:

(a) $h(x) < g(x)$. Since $g(x) \leq \sup\limits_{j \in J} h_j(x)$, then there exists $j = j(x)$ such that $h(x) < h_j(x)$. Take r satisfying $h(x) < r < h_j(x)$. Since h is continuous and h_j is lower semicontinuous, then there exist open neighbourhoods $U(x)$ and $V(x)$ of x such that $h(y) < r$ for any $y \in U(x)$ and $r < h_j(y)$ for any $y \in V(x)$. Let $W(x) = U(x) \cap V(x)$. Then

$$f(y) \leq h(y) \leq h_j(y) \text{ for any } y \in W(x).$$

(b) $h(x) = g(x) = 0$. It follows that $x \in g^{-1}(0)$. By the condition (2) we have $x \in \mathrm{Int} f^{-1}(0)$, and then there exists an open neighbourhood $W(x)$ of x such that $W(x) \subset f^{-1}(0)$. If $y \in W(x)$, then $f(y) \leq h_j(y)$ for any $j \in J$. Therefore, in any case we have an open cover $\{W(x) : x \in X\}$ of X satisfying that for any $x \in X$, there exists $j = j(x)$ such that

$$f(y) \leq h_j(y) \text{ for any } y \in W(x).$$

It follows from the compactness of X that there exists a finite subset $\{x_1, x_2, \cdots, x_n\}$ of X such that

$$X = \bigcup_{i=1}^{n} W(x_i).$$

Since $\{h_j\}_{j \in J}$ is a directed set, then there exists $k \in J$ such that

$$h_{j(x_1)} \leq h_k, \cdots, h_{j(x_n)} \leq h_k.$$

It is obvious that for any $y \in X$, there exists i with $1 \leq i \leq n$ such that $y \in W(x_i)$, which implies that

$$f(y) \leq h_{j(x_i)}(y) \leq h_k(y).$$

It follows that $f \leq h_k$. Hence $f \ll g$ holds.

(1)⇒(2) By the correct part of [1], Proposition I-1.21.1, it suffices to prove that $g^{-1}(0) \subset \mathrm{Int} f^{-1}(0)$. Let

$$G = \{s\chi_U : 0 < s \le 1, U \text{ is open in } X, s < g(x) \text{ for all } x \in U^-\},$$

where χ_U denotes the characteristic function of U, U^- the closure of U. It is easy to prove that $g = \sup G$. Since $f \ll g$, then there exist $s_1\chi_{U_1}, \cdots, s_n\chi_{U_n} \in G$ such that

$$f \le \sup_{1 \le j \le n} s_j \chi_{U_j}.$$

Let $x_0 \in g^{-1}(0)$. Obviously, $x_0 \notin U_j^-$ for any $j = 1, \cdots, n$. Let $U = X - \bigcup_{j=1}^{n} U_j^-$. Then U is open in X and $x_0 \in U$. If $u \in U$, then $u \notin U_j^-$, and therefore $\chi_{U_j}(u) = 0$ for any $j = 1, \cdots, n$. It follows that $f(u) = 0$. Hence $U \subset f^{-1}(0)$. Finally we have

$$g^{-1}(0) \subset \mathrm{Int} f^{-1}(0).$$

References:

[1] GIERZ G. et al. *A Compendium of Continuous Lattices* [M]. Springer-Verlag, 1980.

LSC(X, R*) 上 Way-Below 关系的一个注

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摘 要: 本文指出文献 [1] 中关于下半连续函数格 LSC(X, R*) 上 Way-below 关系的一个结论是错误的，并且给出一个修正的命题及其证明.