Journal of Mathematical Research & Exposition
Vol.19, No.4, 649-653, November 1999

Solution to the Yang-Baxter Equation and Quantum Yang-Baxter H-comodules *

LI Li-bin[1], *WEI Jun-chao*[2]
(1. Dept. of Math., University of Science and Technology of China, Heifei 230026;
2. Dept. of Math., Yangzhou University, Yangzhou 225009)

Abstract: Let H be an arbitrary Hopf algebra over a field k. In this paper, at first we deal with the relationship between solutions to the Yang-Baxter equation and quantum Yang-Baxter H-comodules; then we use the results to give a solution to the Yang-Baxter equation over H.

Key words: Yang-Baxter equation; quantum Yang-Baxter H-comodule; cobraided Hopf algebra.

Classification: AMS(1991) 16W20/CLC O157.1

Document code: A **Article ID:** 1000-341X(1999) 04-0649-05

1. Preliminaries

Throughout this paper, we work over a fixed field k. Let H be a Hopf algebra (bialgebra) with coalgebra structure $\triangle$, counit ε and antipode S. We write $\triangle(h) = \sum h_{(1)} \otimes h_{(2)}$ for $h \in H$. Unless otherwise stated, all maps are k-linear, $\otimes$ means $\otimes_k$, Hom = Hom$_k$, all modules are unital, all H-comodules are left. Our basic reference on Hopf algebra is Sweedler's book [6].

Let H be a bialgebra over k, H is called a cobraided bialgebra, if there exists a bilinear form map r: $H \otimes H \to k, x \otimes y \mapsto r(x \otimes y)$, for all $x, y, z \in H$, satisfying the following conditions:

r is convolution invertible, that is, there exists a bilinear form $\bar{r} : H \otimes H \to R$, such that

$$\sum r(x_{(1)} \otimes y_{(1)})\bar{r}(x_{(2)} \otimes y_{(2)}) = \sum \bar{r}(x_{(1)} \otimes y_{(1)})r(x_{(2)} \otimes y_{(2)}) = \varepsilon(x)\varepsilon(y); \tag{1.1}$$

$$\sum r(x_{(1)} \otimes y_{(1)})x_{(2)}y_{(2)} = \sum y_{(1)}x_{(1)}r(x_{(2)} \otimes y_{(2)}); \tag{1.2}$$

$$r(xy \otimes z) = \sum r(x \otimes z_{(1)})r(y \otimes z_{(2)}); \tag{1.3}$$

*Received date: 1996-01-15
Biography: LI Li-bin (1964-), male, born in Yangzhou county, Jiangsu province. Ph.D, currently an associate professor.

$$r(x \otimes yz) = \sum r(x_{(1)} \otimes z) r(x_{(2)} \otimes y). \tag{1.4}$$

The bilinear form r is called the universal R-form of H. A cobraided bialgebra is also called a coquasitriangular bialgebra[3].

Suppose that V is a finite dimensional vector space over k. A linear map $c \in \mathrm{End}(V \otimes V)$ is said to be a solution of the Yang-baxter equation, if c satisfies condition

$$(c \otimes \mathrm{id}_V)(\mathrm{id}_V \otimes c)(c \otimes \mathrm{id}_V) = (\mathrm{id}_V \otimes c)(c \otimes \mathrm{id}_V)(\mathrm{id}_V \otimes c). \tag{1.5}$$

For general theory of algebraic aspects of the Yang-Baxter equation, we refer reader to see [1,2,4].

Now let (H,r) be a cobraided bialgebra, r be the universal R-form, V be an H-comodule with the comodule struction map $\triangle_V : V \to H \otimes V$, $\triangle_V(v) = \Sigma v_{(-1)} \otimes v_{(0)}$. It is well known that the map $c^r_{V,V} : V \otimes V \to V \otimes V$, $c^r_{V,V}(v \otimes s) = \Sigma r(s_{(-1)} \otimes v_{(-1)}) s_{(0)} \otimes v_{(0)}$ is a solution of the Yang-Baxter equation.

Suppose that V is a finite dimensional vector space, $c \in \mathrm{End}(V \otimes V)$ is a solution to the Yang-Baxter equation. By FRT construction[5], we know that there exists a cobraided bialgebra (H,r) such that V is an H-comodule and $c = c^r_{V,V}$.

In this present paper, we first give the definition of quantum Yang-Baxter H-comodule, present the relationship between a solution of the Yang-Baxter equation and quantum Yang-Baxter H-comodule; as a application, we give a solution of the Yang-Baxter equation over Hopf algebra H.

2. Quantum Yang-Baxter H-comodule and solution of the Yang-Baxter equation

Definition 2.1 *Let $(V, \cdot, \triangle_V)$ be an H-module, H-comodule. We say that V is a quantum Yang-Baxter H-comodule if the following condition holds, for any $h \in H$, $v \in V$*

$$\sum h_{(1)} v_{(-1)} \otimes h_{(2)} \cdot v_{(0)} = \sum (h_{(1)} \cdot v)_{(-1)} h_{(2)} \otimes (h_{(1)} \cdot v)_{(0)}. \tag{2.1}$$

Theorem 2.2 *Suppose that (H,r) is a cobraided Hopf algebra, $(V, \triangle_V)$ is an H-comodule. Define an action of H on V via $h \cdot v = \sum r(v_{(-1)} \otimes h) v_{(0)}, h \in H, v \in V$. Then we have*

(1) $(V, \cdot, \triangle_V)$ is a quantum Yang-Baxter H-comodule.

(2) $c^r_{V,V}(v \otimes s) = \sum v_{(-1)} \cdot s \otimes v_{(0)}, v, s \in V$.

Proof (1) We first prove that V is an H-module under the action. Let $h, g \in H$, $v \in V$. Notice that $r(1 \otimes h) = r(h \otimes 1) = \varepsilon(h)$, therefore $1 \cdot v = \sum r(v_{(-1)} \otimes 1) v_{(0)} = \sum \varepsilon(v_{(-1)}) v_{(0)} = v$, and $h \cdot (g \cdot v) = \sum r(v_{(-2)} \otimes g) r(v_{(-1)} \otimes h) v_{(0)} = \sum r(v_{(-1)} \otimes hg) v_{(0)} = (hg) \cdot v$. It follows that V is an H-module. Second, we prove that V satisfies (2.1). Indeed

$$\sum (h_{(1)} \cdot v)_{(-1)} h_{(2)} \otimes (h_{(1)} \cdot v)_{(0)} = \sum r(v_{(-2)} \otimes h_{(1)}) v_{(-1)} h_{(2)} \otimes v_{(0)}$$
$$= \sum h_{(1)} v_{(-2)} r(v_{(-1)} \otimes h_{(2)}) \otimes v_{(0)} = \sum h_{(1)} v_{(-1)} \otimes h_{(2)} \cdot v_{(0)}.$$

Hence $(V, \cdot, \triangle_V)$ is a quantum Yang-Baxter H-comodule.

(2) For $v, s \in V$,as $c^r_{V,V}(v \otimes s) = \sum r(s_{(-1)} \otimes v_{(-1)})s_{(0)} \otimes v_{(0)}$, by the action of H on V, the above equals $\sum v_{(-1)} \cdot s \otimes v_{(0)}$.

Suppose further that V, W are H-modules, H-comodules. In usual way, we take $V \otimes W$ as an H-module, H-comodule via $h \cdot (v \otimes w) = \sum h_{(1)} \cdot v \otimes h_{(2)}$, and $\triangle_{V \otimes W}(v \otimes w) = \sum v_{(-1)}w_{(-1)} \otimes v_{(0)} \otimes w_{(0)}, v \in V, w \in W$. Take $c_{V,W}$: $V \otimes W \to W \otimes V, v \otimes w \mapsto v_{(-1)} \cdot w \otimes v_{(0)}$. Then we have

Theorem 2.3 *Suppose that V is a finite dimensional vector space, c is a solution to the Yang-Baxter equation. Then there exists a cobraided bialgebra (H, r) such that V is a quantum Yang-Baxter H-comodule, and $c = c_{V,W} = c^r_{V,W}$.*

Proof It follows easily from FRT construction and Theorem 2.2.

Theorem 2.4 *Suppose that H is a Hopf algebra with invertible antipode S, V, W are H-modules, H-comodules. Then*

(1) $c_{V,W}$ is invertible.

(2) If W is a quantum Yang-Baxter H-comodule, then $c_{V,W}$ is an H-comodule morphism.

(3) If V is a quantum Yang-Baxter H-comodule, then $c_{V,W}$ is an H-module morphism.

Proof (1) Take $t_{W,V}$: $W \otimes V \to V \otimes W, w \otimes v \mapsto \sum v_{(0)} \otimes S^{-1}(v_{(-1)}) \cdot w$, then we have

$$t_{W,V}c_{V,W}(v \otimes w) = \sum t_{w,v}(v_{(-1)} \cdot w \otimes v_{(0)}) = \sum v_{(0)} \otimes S(v_{(-1)}) \cdot (v_{(-2)} \cdot w)$$
$$= \sum v_{(0)} \otimes \varepsilon(v_{(-1)})w = v \otimes w.$$

Hence $c_{V,W}$ is left invertible. It is similar to prove that $c_{V,W}$ is right invertible.

(2) Since

$$(\mathrm{id}_H \otimes c_{V,W})\triangle_{V \otimes W}(v \otimes w) = \sum v_{(-2)}w_{(-1)} \otimes v_{(-1)} \cdot w_{(0)} \otimes v_{(0)}$$
$$= \sum (v_{(-2)} \cdot w)_{(-1)}v_{(-1)} \otimes (v_{(-2)} \cdot w)_{(0)} \otimes v_{(0)} = \triangle_{W \otimes V}c_{V,W}(v \otimes w),$$

it follows that $\triangle_{W \otimes V}c_{V,W} = (\mathrm{id}_H \otimes c_{V,W})\triangle_{V \otimes W}$, so $c_{V,W}$ is an H-comodule morphism.

(3) For any $h \in H, v \in V, w \in W$, since

$$c_{V,W}(h \cdot (v \otimes w)) = \sum (h_{(1)} \cdot v)_{(-1)} \cdot (h_{(2)} \cdot w) \otimes (h_{(1)} \cdot v_{(0)})$$
$$= \sum ((h_{(1)} \cdot v)_{(-1)}h_{(2)}) \cdot w \otimes (h_{(1)} \cdot v_{(0)}) = \sum (h_{(1)}v_{(-1)}) \cdot w \otimes h_{(2)} \cdot v_{(0)}$$
$$= h \cdot c_{V,W}(h \cdot (v \otimes w)).$$

It follows that $c_{V,W}$ is an H-module morphism.

Especially, let H be a Hopf algebra. We regard H as a regular H-module, H-comodule. Then we have

Corollary 2.5 *Let H be a Hopf algebra, V be an H-module, H-module. Then the following are equivalent:*

(1) V is a quantum Yang-Baxter H-comodule.

(2) $c_{H,V}$ is an H-comodule morphism.

(3) $c_{V,H}$ *is an* H*-module morphism.*

Proof (1)⇒ (2) and (1)⇒ (3) follow from theorem 2.4.

(2)⇒ (1) Since $c_{H,V}$ is an H-module morphism, it follows that $(\mathrm{id}_H \otimes c_{H,V})\triangle_{H\otimes V} = \triangle_{V\otimes H} c_{H,V}$. Hence, for $h \in H, v \in V$

$$\sum h_{(1)}v_{(-1)} \otimes h_{(2)} \cdot v_{(0)} \otimes h_{(3)} = \sum (h_{(1)} \cdot v)_{(-1)} h_{(2)} \otimes (h_{(1)} \cdot v)_{(0)} \otimes h_{(3)}.$$

Apply ε to the third tensor idems of the equality above, then we obtain (2.1), i.e., V is a quantum Yang-Baxter H-comodule.

(3)⇒ (1) Since $c_{V,H}$ is an H-module morphism, it follws that for $h, g \in H, v \in V$ we have $c_{V,H}(h \cdot (v \otimes g)) = h \cdot (c_{V,H}(v \otimes g))$, i.e.,

$$\sum((h_{(1)} \cdot v)_{(-1)} h_{(2)})g \otimes (h_{(1)} \cdot v)_{(0)} = \sum (h_{(1)} v_{(-1)})g \otimes h_{(2)} \cdot v_{(0)},$$

take $g = 1$, we obtain (2.1). Hence we completes the proof.

Theorem 2.6 *Suppose* V, W, U *are* H*-modules,* H*-comodules, then we have*

(1) $c_{V\otimes W,U} = (c_{V,U} \otimes \mathrm{id}_W)(\mathrm{id}_V \otimes c_{W,U})$, *and* $c_{V,W\otimes U} = (\mathrm{id}_W \otimes c_{V,U})(c_{V,W} \otimes \mathrm{id}_U)$.

(2) If W *is a quantum Yang-Baxter* H*-comodule, then*

$$(\mathrm{id}_U \otimes c_{V,W})(c_{V,U} \otimes \mathrm{id}_W)(\mathrm{id}_V \otimes c_{W,U}) = (c_{W,U} \otimes \mathrm{id}_V)(\mathrm{id}_W \otimes c_{V,U})(c_{V,W} \otimes \mathrm{id}_U).$$

(3) If V *is a quantum Yang-Baxter* H*-comodule, then* $c_{V,V}$ *is a solution of the Yang-Baxter equation.*

Proof (1) For any $v \in V, u \in U, w \in W$, we have

$$\begin{aligned}&(c_{V,U} \otimes \mathrm{id}_W)(\mathrm{id}_V \otimes c_{W,U})(v \otimes w \otimes u) = (c_{V,U} \otimes \mathrm{id}_W)(\sum v \otimes w_{(-1)} \cdot u \otimes w_{(0)})\\ &= \sum (v_{(-1)} w_{(-1)}) \cdot u \otimes v_{(0)} \otimes w_{(0)}) = c_{V,W\otimes U}(v \otimes w \otimes u).\end{aligned}$$

Hence the first equality holds. Simlarly, we can prove the second.

(2) Since

$$\begin{aligned}&(\mathrm{id}_U \otimes c_{V,W})(c_{V,U} \otimes \mathrm{id}_W)(\mathrm{id}_V \otimes c_{W,U})(v \otimes w \otimes u)\\ &= \sum (v_{(-2)} w_{(-1)}) \cdot u \otimes v_{(-1)} \cdot w_{(0)} \otimes v_{(0)}\\ &= \sum ((v_{(-2)} \cdot w)_{(-1)} v_{(-1)}) \cdot u \otimes (v_{(-2)} \cdot w)_{(0)} \otimes v_{(0)}\\ &= (c_{W,U} \otimes \mathrm{id}_V)(\sum v_{(-2)} \cdot w \otimes v_{(-1)} \cdot u \otimes v_{(0)})\\ &= (c_{W,U} \otimes \mathrm{id}_V)(\mathrm{id}_W \otimes c_{V,U})(c_{V,W} \otimes \mathrm{id}_U)(v \otimes w \otimes u).\end{aligned}$$

Hence (2) holds.

(3) Take $V = W = U$ and use Theorem 2.4(1).

3. Applications

Suppose that H is a Hopf algebra. It is well known that H is an H-module algebra via $h \cdot g = \sum h_{(1)} g S(h_{(2)})$, on the other hand, H is a regular H-comodule. Moreover we have

Theorem 3.1 *(1) H is a quantum Yang-Baxter H-comodule.*

(2) $c_{H,H}(h\otimes g)=\sum h_{(1)}gS(h_{(2)})\otimes h_{(3)}$ is a solution to the Yang-Baxter equation.

Proof (1) Take $h,g\in H$, on the one hand

$$\sum h_{(1)}g_{(1)}\otimes h_{(2)}\cdot g_{(2)}=\sum h_{(1)}g_{(1)}\otimes h_{(2)}g_{(2)}S(h_{(3)}).$$

On the other hand

$$\begin{aligned}&\sum(h_{(1)}\cdot g)_{(1)}h_{(2)}\otimes(h_{(1)}\cdot g)_{(2)}=\sum(h_{(1)}gS(h_{(2)}))_{(1)}h_{(3)}\otimes(h_{(1)}gS(h_{(2)}))_{(2)}\\&=\sum h_{(1)}g_{(1)}S(h_{(4)})h_{(5)}\otimes h_{(2)}g_{(2)}S(h_{(3)})=\sum h_{(1)}g_{(1)}\varepsilon(h_{(4)})\otimes h_{(2)}g_{(2)}S(h_{(3)})\\&=\sum h_{(1)}g_{(1)}\otimes h_{(2)}g_{(2)}S(h_{(3)}).\end{aligned}$$

It follows that H is a quantum Yang-Baxter H-comodule.

(2) It follows from Theorem 2.6(3).

Corollary 3.2 *Suppose that G is a finite group, $G=\{g_1,g_2,\cdots,g_n\}$, then $c\in\mathrm{End}(kG\otimes kG)$, $c(g_i\otimes g_j)=g_ig_jg_i^{-1}\otimes g_{(i)}$, $(i,j=1,2,\cdots,n)$ is a solution to the Yang-Baxter equation.*

Acknowledgements The authors would like to thank Prof Cai Chuanren and Chen Huixiang for their comments.

References:

[1] LAMBE L A, RADFORD D E. *Algebraic aspects of the quantum Yang-Baxter equation* [J]. J. Alg., 1992, **154**: 228–288.

[2] RADFORD D E. *Solutions to the quantum Yang-Baxter equation and Drinfel'd double* [J]. J. Alg., 1993, **161**: 20–33.

[3] MAJID S. *Quasitriangular Hopf algebra and Yang-Baxter equation* [J]. Internat. J. Modern Phys., 1990, **A5**: 1–91.

[4] DRINFELD V G. *Hopf algebras and the quantum Yang-Baxter equation* [J]. Soviet Math. Dok., 1985, **32**: 254–258.

[5] KASSEL C. *Quantum Groups* [M]. Springer-Verlag, New York, 1995.

[6] SWEEDLER M C. *Hopf Algebra* [M]. Benjamin, New York, 1969.

Yang-Baxter 方程的解与量子 Yang-Baxter H- 余模

李 立 斌[1]， 魏 俊 潮[2]

(1. 中国科学技术大学数学系，合肥 230026;

2. 扬州大学数学系，扬州 225002)

摘 要: 设 H 是域 k 上的 Hopf 代数．本文首先讨论了量子 Yang-Baxter H- 余模与 Yang-Baxter 方程的解的关系；然后作为应用，给出了任意 Hopf 代数上 Yang-Baxter 方程的一个解．