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A Generalized Markov Inequality *

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Abstract: This paper gives a generalized Markov inequality $\int_{-1}^{1} f(|P'|)\mathrm{d}x \le \int_{-1}^{1} f(\|P\|T_n')\mathrm{d}x$ for every polynomial P of degree at most n provided that f' is continuous and strictly increasing on $[0,\infty)$, where $\|\cdot\|$ denotes the uniform norm and T_n stands for the n-th Chebyshev polynomial of the first kind.

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1. Introduction

Denote by $\mathbf{P}_n$ the set of polynomials of degree at most n and by T_n the n-th Chebyshev polynomial of the first kind. Let $\|\cdot\|$ stand for the uniform norm and write

$$F(P) := \int_{-1}^{1} f[|P(x)|]\mathrm{d}x.$$

This paper deals with a generalized Markov inequality

$$F(P') \le F(\|P\|T_n'), \quad P \in \mathbf{P}_n. \tag{1.1}$$

Several authors studied this inequality. In 1982, using a variational approach, Bojanov obtained an extension of the Markov inequality:

Theorem A[2] *Let $1 < p < \infty$ and $f(x) = x^p$. Then the inequality (1.1) holds and the equality is attained if and only if $P = \pm\|P\|T_n$.*

Meanwhile, in 1982, using a variational approach and a technique different from [2], Bojanov solved a conjecture proposed by Erdös in 1939 in [4]:

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Theorem B[3] *Let $f(x)=(1+x^2)^{1/2}$. Then the inequality (1.1) holds and the equality is attaiend if and only if $P=\pm\|P\|T_n$.*

In 1984, using the idea of [2], Zhou extended the result of Theorem A:

Theorem C[5] *Let f satisfy the conditions:*

(a) $f(x)$ is increasing on $[0,\infty)$;

(b) $f(x)$ is strictly convex on $[0,\infty)$;

(c) $f(0)=0$.

Then the inequality (1.1) holds.

But, we observe that Condition (c) of Theorem C restricts its use. Following the idea of Bojanov in [3] and using Lagrange's method of multipliers, we will establish a generalized Markov inequality (1.1), dropping Condition (c) in this paper. That is the following

Theorem 1 *Assume that the function f is continuously differentiable on $[0,\infty)$ and satisfies the condition:*

$$f(x)-xf'(x)<f(0)<f(x),\quad x\in(0,\infty). \tag{1.2}$$

Then (1.1) holds and the equality occurs if and only if $P=\pm\|P\|T_n$.

As a consequence of Theorem 1 we state

Theorem 2 *Assume that the function f' is continuous and is strictly increasing on $[0,\infty)$. Then (1.1) holds and the equality occurs if and only if $P=\pm\|P\|T_n$.*

This result also provides a characterization of the Chebyshev polynomials T_n. Now by Theorem 2 it admits the following extension:

$$f(x)=(a+bx^p)^{1/p},\quad a,b>0,\quad 1<p<\infty$$

since

$$f'(x)=b^{1/p}\left(\frac{bx^p}{a+bx^p}\right)^{(p-1)/p}$$

satisfies the conditions of Theorem 2.

2. Auxiliary Lemmas

Let $M>0$ be a fixed number. Following Bojanov[2] we define the sets $\Omega_{\mathbf{m}}\subset\omega_{\mathbf{m}}\subset\mathbf{P}_n$ as follows. $Q\in\omega_{\mathbf{m}}$ if and only if $Q\in\mathbf{P}_n$ has exactly $m-1$ extremal points $x_i=x_i(Q)$, $i=1,\ldots,m-1$ in $(-1,1)$,

$$-1=x_0<x_1<\ldots<x_{m-1}<x_m=1; \tag{2.1}$$

$Q\in\Omega_{\mathbf{m}}$ if and only if $Q\in\omega_{\mathbf{m}}$ and $|Q(x_i(Q))|=M, i=0,1,\cdots,m$. For convenience let $r(x):=g(x)-2xg'(x)$ and

$$G(P):=\int_{-1}^{1}g[P(x)^2]\mathrm{d}x.$$

Then we have

Lemma 1 *Assume that the function $g \in C[0,\infty)$ satisfies the conditions:*

$$xg'(x^2) \in C[0,\infty); \quad r(x) < g(0) < g(x), \quad x \in (0,\infty). \tag{2.2}$$

If $P \in \omega_{\mathbf{m}}$ is a solution of the programming problem

$$\max_{Q \in \mathbf{P}_n} G(Q') \tag{2.3}$$

subject to

$$\|Q\| \le M, \tag{2.4}$$

then $P \in \Omega_{\mathbf{m}}$.

Proof For simplicity write $y_i := x_i(P), i = 0,1,\cdots,m$. Then we have

Claim 1. There exists a vector $\lambda = (\lambda_0, ..., \lambda_m, 1)$, $\lambda \ge 0$, such that

$$\int_{-1}^{1} g'[P'(x)^2]P'(x)R'(x)\mathrm{d}x - \sum_{i=0}^{m} \lambda_i P(y_i)R(y_i) = 0, \quad \forall R \in \mathbf{P}_n, \tag{2.5}$$

$$\lambda_i[P(y_i)^2 - M^2] = 0, \quad i = 0,1,\cdots,m. \tag{2.6}$$

To prove this claim we need a basic result, in which

$$\nabla h(\mathbf{y}) = \left(\frac{\partial h(\mathbf{x})}{\partial x_1}, \cdots, \frac{\partial h(\mathbf{x})}{\partial x_n}\right)\Bigg|_{\mathbf{x}=\mathbf{y}}.$$

Theorem D[1,Theorem3.4] *Assume that $g_0, g_1, \cdots, g_m$ are continuously differentiable on an open set $S \subset \mathbf{R}^n$. If $\mathbf{y} \in S$ is a solution of the problem to minimize $g_0(\mathbf{x})$ subject to $\mathbf{x} \in S$ and $g_i(\mathbf{x}) \le 0$, $i = 1,2,\cdots,m$, then there exists a vector $\lambda = (\lambda_0, \lambda_1, \cdots, \lambda_m) \ne 0, \lambda \ge 0$, such that*

$$\sum_{i=0}^{m} \lambda_i \nabla g_i(\mathbf{y}) = 0$$

and

$$\lambda_i g_i(\mathbf{y}) = 0, \quad i = 1,2,\cdots,m.$$

Let $Q \in \omega_m$ and $Q(x) = \sum_{j=0}^{n} a_j x^j$. Then (2.4) is equivalent to

$$Q(x_i)^2 \le M^2, \quad i = 0,1,\cdots,m \tag{2.7}$$

and P is also a solution of the programming problem (2.3) subject to (2.7) with $Q \in \omega_{\mathbf{m}}$. By Theorem D there exists a nonzero vector $\lambda = (\lambda_0, \cdots, \lambda_{m+1})$, $\lambda \ge 0$, such that (2.6) holds and

$$\left[-\lambda_{m+1}\frac{\partial G(Q')}{\partial a_j} + 2\sum_{i=0}^{m} \lambda_i Q(x_i)\frac{\partial Q(x_i)}{\partial a_j}\right]\Bigg|_{Q=P} = 0, \quad j = 0,1,\cdots,n. \tag{2.8}$$

It is easy to calculate that

$$\frac{\partial G(Q')}{\partial a_j} = 2j \int_{-1}^{1} g'[Q'(x)^2]Q'(x)x^{j-1}\mathrm{d}x.$$

Meanwhile we point out that

$$\frac{\partial Q(x_i)}{\partial a_j} = (x_i)^j. \tag{2.9}$$

In fact, it is trivial for $i = 0$ and $i = m$. For $1 \le i \le m-1$ we see

$$\frac{\partial Q(x_i)}{\partial a_j} = (x_i)^j + Q'(x_i)\frac{\partial x_i(Q)}{\partial a_j}.$$

Since x_i is an extremal point of Q, there is an index $k \ge 1$ such that

$$Q'(x_i) = \cdots = Q^{(2k-1)}(x_i) = 0, \quad Q^{(2k)}(x_i) \ne 0,$$

and it suffices to show $|\frac{\partial x_i(Q)}{\partial a_j}| < \infty$. But this fact follows from the equation

$$\frac{\partial Q^{(2k-1)}(x_i)}{\partial a_j} + Q^{(2k)}(x_i)\frac{\partial x_i(Q)}{\partial a_j} = 0,$$

which may be obtained by partially differentiating the relation $Q^{(2k-1)}(x_i) = 0$. This proves (2.9). Then (2.8) becomes

$$-\lambda_{m+1} j \int_{-1}^{1} g'[P'(x)^2]P'(x)x^{j-1}\mathrm{d}x + \sum_{i=0}^{m} \lambda_i P(y_i)(y_i)^j = 0, \quad j = 0, 1, \cdots, n.$$

Let $R(x) = \sum_{j=0}^{n} c_j x^j$. Multiplying the above j-th equation with c_j and summing the resulting equations gives

$$-\lambda_{m+1} \int_{-1}^{1} g'[P'(x)^2]P'(x)R'(x)\mathrm{d}x + \sum_{i=0}^{m} \lambda_i P(y_i)R(y_i) = 0. \tag{2.10}$$

In particular, setting $R = P$ yields

$$-\lambda_{m+1} \int_{-1}^{1} g'[P'(x)^2]P'(x)^2\mathrm{d}x + \sum_{i=0}^{m} \lambda_i P(y_i)^2 = 0.$$

This equation means $\lambda_{m+1} \ne 0$, for otherwise, taking into account (2.6), we conclude $\sum_{i=0}^{m} \lambda_i = 0$, i.e., $\lambda = 0$ (because $\lambda \ge 0$), contradicting $\lambda \ne 0$. So we can suppose without loss of generality that $\lambda_{m+1} = 1$ and hence (2.5) follows from (2.10). This proves Claim 1.

Claim 2. We have

$$\lambda_i > 0, \quad i = 0, 1, \cdots, m. \tag{2.11}$$

To find a formula for λ_i let $L(x) := (x^2-1)P'(x)$ and

$$L_i(x) = \frac{L(x)}{x-y_i}, \quad i = 0, 1, \cdots, m.$$

Substituting $R = L_i$ into (2.5) gives

$$\lambda_i P(y_i)L'(y_i) = \int_{-1}^{1} g'[P'(x)^2]P'(x)L_i'(x)\mathrm{d}x, \quad i = 0, 1, \cdots, m. \tag{2.12}$$

It is particularly simple to prove (2.11) for $i=0$ and $i=m$; we do this for $i=m$. In this case by integration by parts (2.12) gives

$$\begin{aligned}\lambda_m P(1)L'(1) &= \int_{-1}^{1} g'[P'(x)^2]P'(x)[P'(x)+(x+1)P''(x)]\mathrm{d}x \\ &= \int_{-1}^{1} g'[P'(x)^2]P'(x)^2\, dx + \frac{1}{2}\int_{-1}^{1}(x+1)\mathrm{d}g[P'(x)^2] \\ &= g[P'(1)^2] - \frac{1}{2}\int_{-1}^{1} r[P'(x)^2]\mathrm{d}x.\end{aligned}$$

By (2.2)

$$\frac{1}{2}\int_{-1}^{1} r[P'(x)^2]\mathrm{d}x < g(0) \le g[P'(1)^2].$$

Thus $\lambda_m P(1)L'(1) > 0$ and hence $\lambda_m > 0$.

Now assume $1 \le i \le m-1$. Put $I(d) = [-1, y_i - d] \cup [y_i + d, 1]$ with $0 < d < \min_{0\le i\le m-1}(y_{i+1} - y_i)$, $q(x) := (x^2-1)/(x-y_i)$, and

$$\Lambda(d) := \int_{I(d)} g'[P'(x)^2]P'(x)L_i'(x)\mathrm{d}x.$$

Again by partial integration

$$\begin{aligned}2\Lambda(d) + \int_{I(d)} r[P'(x)^2]q'(x)\mathrm{d}x &= \int_{I(d)} \left\{g[P'(x)^2]q'(x) + 2g'[P'(x)^2]P'(x)P''(x)q(x)\right\}\mathrm{d}x \\ &= g[P'(y_i-d)^2]q(y_i-d) - g[P'(y_i+d)^2]q(y_i+d).\end{aligned}$$

Thus

$$\begin{aligned}2\Lambda(d) = &\left\{g[P'(y_i-d)^2]q(y_i-d) - \int_{-1}^{y_i-d} r[P'(x)^2]q'(x)\mathrm{d}x\right\} + \\ &\left\{-g[P'(y_i+d)^2]q(y_i+d) - \int_{y_i+d}^{1} r[P'(x)^2]q'(x)\mathrm{d}x\right\} \\ := &\ \Lambda_1(d) + \Lambda_2(d).\end{aligned} \tag{2.13}$$

To estimate $\Lambda_1(d)$ we break the integral into two parts over $[-1, y_{i-1}]$ and $[y_{i-1}, y_i - d]$. For the first integral with $i > 1$, noting that $q'(x) = 1 + (1-y_i^2)/(x-y_i)^2 > 0$ on $I(d)$, by the mean-value theorem for integrals, there is a point $\xi \in [-1, y_{i-1}]$ such that

$$\int_{-1}^{y_{i-1}} r[P'(x)^2]q'(x)\mathrm{d}x = r[P'(\xi_1)^2]\int_{-1}^{y_{i-1}} q'(x)\mathrm{d}x = r[P'(\xi_1)^2]q(y_{i-1}). \tag{2.14}$$

This formula remains true for $i=1$, because in this case each term in (2.14) is zero. Moreover, by (2.2)

$$\int_{y_{i-1}}^{y_i-d} r[P'(x)^2]q'(x)\mathrm{d}x < g(0)\int_{y_{i-1}}^{y_i-d} q'(x)\mathrm{d}x = g(0)\left[q(y_i-d)-q(y_{i-1})\right]. \qquad (2.15)$$

Then (2.13)–(2.15) by (2.2) leads to

$$\begin{aligned}\Lambda_1(d) &> \{g(0)-r[P'(\xi_1)^2]\}q(y_{i-1})+\{g[P'(y_i-d)^2]-g(0)\}q(y_i-d)\\ &\geq \{g(0)-r[P'(\xi_1)^2]\}q(y_{i-1}).\end{aligned}$$

Similarly we can get

$$\Lambda_2(d) > -\{g(0)-r[P'(\xi_2)^2]\}q(y_{i+1}),$$

where $\xi_2 \in [y_{i+1},1]$. Finally by (2.13) and (2.2)

$$\begin{aligned}&\lambda_i P(y_i)L'(y_i) = \lim_{d\to 0}\Lambda(d)\\ &\geq \frac{1}{2}\left\{g(0)-r[P'(\xi_1)^2]\right\}\left[\frac{1-y_{i-1}^2}{y_i-y_{i-1}}\right]+\frac{1}{2}\left\{g(0)-r[P'(\xi_2)^2]\right\}\left[\frac{1-y_{i+1}^2}{y_{i+1}-y_i}\right] > 0.\end{aligned}$$

Hence $\lambda_i > 0$. This proves (2.11). So by (2.6) we obtain $P(y_i)^2 = M^2$, $i=0,1,\cdots,m$, which means $P\in\Omega_{\mathbf{m}}$. □

Lemma 2 *Let g satisfy (2.2). If $P\in\Omega_{\mathbf{m}}$ then*

$$G(P') \leq G(\|P\|T_n') \qquad (2.16)$$

holds and the equality occurs if and only if $P=\pm\|P\|T_n$.

Proof The proof follows the idea of [3]. Note first by (2.2) that the inequality $y\geq z>0$ implies

$$yg(y^{-2})-zg(z^{-2}) \leq g(0)(y-z). \qquad (2.17)$$

In fact, by the mean-value theorem for derivatives it follows from (2.2) that

$$yg(y^{-2})-zg(z^{-2}) = [xg(x^{-2})]'_{x=\xi}(y-z) = r(\xi^{-2})(y-z) \leq g(0)(y-z).$$

Suppose that $[-1,1]=\cup_{i=0}^m I_i$ is the partition of $[-1,1]$ induced by P and the intevals $I=[z_1,z_2]$ and $I^*=[z_1^*,z_2^*]$ are corresponding in the sense of [3]. We need the following

Lemma A[3,Lemma 2] *Suppose that $P\in\Omega_{\mathbf{m}}, y\in(-M,M)$, and $k\in\{0,1,...,m\}$. Let ξ and η satisfy the conditions*

$$\xi\in I_k^*,\quad MT_n(\xi)=y,\quad \eta\in I_k,\quad P(\eta)=y.$$

Then $|P'(\eta)|\leq|MT_n'(\xi)|$.

Let $u(y)$ and $v(y)$ be the inverse functions of $P(x)$ and $MT_n(x)$ in I and I^*, respectively. Since

$$P'(\eta)=\frac{1}{u'(y)},\quad MT_n'(\xi)=\frac{1}{v'(y)},\quad y\in(-M,M), \qquad (2.18)$$

by Lemma A $|v'(y)| \le |u'(y)|$ holds for all $y \in (-M, M)$. Then (2.17) leads to

$$\int_{-M}^{M} |u'(y)| g[u'(y)^{-2}] \mathrm{d}y \le \int_{-M}^{M} |v'(y)| g[v'(y)^{-2}] \mathrm{d}y + g(0) \left\{ \int_{-M}^{M} |u'(y)| \mathrm{d}y - \int_{-M}^{M} |v'(y)| \mathrm{d}y \right\}.$$

By means of (2.18) the above inequality becomes, noting that both $u'(y)$ and $v'(y)$ do not change sign in $(-M, M)$,

$$\int_{z_1}^{z_2} g[P'(x)^2] \mathrm{d}x \le \int_{z_1^*}^{z_2^*} g[M^2 T_n'(x)^2] \mathrm{d}x + g(0)\{|z_2 - z_1| - |z_2^* - z_1^*|\}.$$

Summing the above inequalities for $I = I_0, \cdots, I_m$ and uing (2.2) yields, in which $|I^*| = z_2^* - z_1^*$,

$$\begin{aligned} G(P') &\le \int_{\cup_{i=0}^m I_i^*} g[M^2 T_n'(x)^2] \mathrm{d}x + g(0)\{2 - |\cup_{i=0}^m I_i^*|\} \\ &= G(MT_n') + g(0)\{2 - |\cup_{i=0}^m I_i^*|\} - \int_{[-1,1]\setminus \cup_{i=0}^m I_i^*} g[M^2 T_n'(x)^2] \mathrm{d}x \\ &\le G(MT_n'). \end{aligned}$$

The equality occurs if and only if $\cup_{i=0}^m I_i^* = [-1, 1]$, which according to the definition of I_i^*'s in [3] means $P = \pm M T_n = \pm \|P\| T_n$ (because $\|P\| = M$ for $P \in \Omega_{\mathbf{m}}$). □

3. Proofs

3.1. Proof of Theorem 1

Since $P \in \mathbf{P}_n$ implies that there is an index $m, 1 \le m \le n$, such that $P \in \omega_{\mathbf{m}}$, according to Lemmas 1 and 2 to prove Theorem 1 it is enough to verify that the function $g(x) = f(x^{1/2})$ satisfies (2.2). In fact, we have that $xg'(x^2) = \frac{1}{2} f'(x)$, which belongs to $C[0, \infty)$. Meanwhile, by (1.2) we have $r(x) = f(x^{1/2}) - x^{1/2} f'(x^{1/2}) < f(0) < f(x^{1/2})$ for $x > 0$, which means $r(x) < g(0) < g(x)$. □

3.2. Proof of Theorem 2

It suffices to verify (1.2). By the mean-value theorem of derivatives for some point $\xi, x > \xi > 0$, we have that $f(x) - f(0) = f'(\xi)x$ and $0 < f'(\xi) < f'(x)$, which means $f(x) - xf'(x) < f(0) < f(x)$. □

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