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The Existence of Periodic Orbits to a Higher Dimensional Autonomous System with a Non-hyperbolic Singular Point *

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Abstract: By means of theory of topological degree in nonlinear functional analysis combining with qualitative analysis in ordinary differential equations, we discuss the existence of nontrivial periodic orbits to a higher dimensional autonomous system with a non-hyperbolic singular point.

Key words: autonomous system; non-hyperbolic singular point; topological degree; index of a singular point.

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1. Introduction

Grasman W.[1] discussed the existence of periodic orbits to a general n-dimension autonomous system,the result of paper [2] showed that Grasman Theorem contains an unnecessary condition. Adding small perturbation to the system [1,2], paper [3] got a similar result. But papers [1–3] all assumed that the singular point is hyperbolic, i.e., the real parts of all eigenvalues of Jacobi matrix for system at a singular point are not zeroes. The purpose of this paper is to investigate the existence of periodic orbits to a n-dimension autonomous system with a non-hyperbolic singular point.

Consider a n-dimension autonomous dynamic system

$$\dot{x} = f(x), \quad x \in R^n, \tag{1}$$

where $f : M(\subset R^n) \longrightarrow R^n, f \in C^1(M), M$ is a positively invariant compact set of system (1).

System (1) define a dynamic flow $A : R_+ \times M \longrightarrow M, (t, x_0) \longrightarrow A(t, x)$,

$$A(t, A(s, x)) = A(t + s, x), A(0, x) = x,$$

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where for any $x_0 \in M, A(t, x_0) = x(t)$ is a solution to system (1) satisfying initial value condition $x(0) = A(0, x_0) = x_0, R_+ = [0, +\infty), t, s \in R_+$.

The scalar form of system (1) is

$$\dot{x}_i = f_i(x_1, \cdots, x_n), \quad 1 \le i \le n. \tag{2}$$

Define the transformation $C : y \longrightarrow x, y \in R^n$ as follows

$$x_1 = y_2 \cos y_1, \quad x_2 = y_2 \sin y_1, \quad x_j = y_j, \quad 3 \le j \le n. \tag{3}$$

System (2) is changed into

$$\dot{y}_i = g_i(y_1, y_2, \cdots, y_n), \quad 1 \le i \le n. \tag{4}$$

Noticing that under the transformation (3),we have

$$\dot{y}_1 = (\dot{x}_2 \cos y_1 - \dot{x}_1 \sin y_1)/y_2, \quad \dot{y}_2 = \dot{x}_1 \cos y_1 + \dot{x}_2 \sin y_1.$$

When $y_2 = 0$,

$$\dot{x}_2 \cos y_1 - \dot{x}_1 \sin y_1 = f_2(0, 0, y_3, \cdots, y_n) \cos y_1 - f_1(0, 0, y_3, \cdots, y_n) \sin y_1,$$

so we assume that $f_i(0, 0, x_3, \cdots, x_n) = 0, i = 1, 2$ such that the value of function $g_1(y_1, y_2, \cdots,$ at $y_2 = 0$ can be defined by means of limit.

For convenience,we introduce the notation $M^C = \{y \in R^n | C(y) \in M\}$, where C is transformation (3).

2. The Main Result

Theorem 1 *Assume that*

(i) M is a positively invariant compact set of (1) and a star-shaped neighborhood of the origin O.

(ii) O is the unique singular point of system (1) in set M. Let $J = f_x(0)$ be Jacobi matrix of $f(x)$ at $x = 0$. 0 is an eigenvalue of J with multiplicity $2r$, where r is a nonnegative integer. Each of the pure imaginary eigenvalues βi $(\beta \neq 0)$ of J satisfies: $2\beta \neq mb$ or $\beta \neq mb$, where m is any integer and $b = [-4f_{12}f_{21} - (f_{11} - f_{22})^2]^{\frac{1}{2}}/2, f_{ij} = \frac{\partial f_i}{\partial x_j}|_{x=0}, i, j = 1, 2$. Among the eigenvalues of J with non-zero real parts, there are $2p$ eigenvalues with positive real parts, the remaining ones have negative real parts.

(iv) system (1) or (2) can be changed into (4) through transformation (3), where $g_i \in C^1(M^C)$ and $g_1 \neq 0$ on M^C.

then there exists at least a nontrivial periodic orbit to system (1) in set M.

In this paper,we use the same numbering way as the theorems of papers [2,3]. The condition (iii) is omitted in Theorem 1 because we have shown that it is unnecessary in paper [2].

3. The proof of main result

Proof Define set $\Gamma_\phi = \{C(y) | y \in M^C, y_1 = \phi \text{ or } y_1 = \pi + \phi\}$. By Lemma 1 [2],$\Gamma_\phi$ is a $n-1$

dimensional star-shaped neighborhood of the origin O. By Lemma 3 [2], we can define a mapping $F:\Gamma_\phi \longrightarrow \Gamma_\phi, x \longrightarrow A(t(y),x)$, where $y \in \{M^C|y_1=\phi \text{ or } y_1=\pi+\phi\}, C(y)=x$, and $F\in C^1(\Gamma_\phi,\Gamma_\phi)$.

For any $x\in\Gamma_\phi$, set $V(x)=F(x)-x, V$ is a vector field on Γ_ϕ. It is clear that $V(0)=0$. Noticing that $V(x^*)=0 \Longleftrightarrow F(x^*)=x^* \Longleftrightarrow A(t(y^*),x^*)=x^*$ if $x^*\neq 0$, then the orbit through x^* is a nontrivial periodic orbit to system (1). Therefore the proof of Theorem 1 is completed if we can prove the existence of x^*. Now we come to establish the existence of x^*.

If there exists a $x^*\in\partial\Gamma_\phi$ such that $V(x^*)=0$, obviously Theorem 1 holds. So we suppose that for any $x\in\partial\Gamma_\phi, V(x)\neq 0$, i.e., $0\notin V(\partial\Gamma_\phi)$. It is divided into the following two cases to prove.

(a) $V^{-1}(0)=\{a|a\in\Gamma_\phi, V(a)=0\}$ is an infinite set. $V(x)=0$ has infinite non-zero solutions in Γ_ϕ, hence there exist infinite nontrivial periodic orbits to system (1). Theorem 1 holds.

(b) $V^{-1}(0)$ is a finite set.Since Γ_ϕ is a star-shaped set and M is a positively invariant set,vector fields $V(x)$ on $\partial\Gamma_\phi$ can not point to the outside of Γ_ϕ along the radial direction.By Lemma 5 [2],the topological degree of mapping V at origin O on Γ_ϕ satisfies $d(V,\Gamma_\phi,0)=(-1)^{n-1}$. It is known from Theorem [4] of index sum of zero points to topological degree that

$$d(V,\Gamma_\phi,0)=\sum_{a\in V^{-1}(0)} i(V,a,0)=i(V,0,0)+i(V,x^*,0)+\cdots, \tag{5}$$

where $i(V,a,0)$ is the index of mapping V at point a.

If we can prove $i(V,0,0)=(-1)^{n-2k}$ (k is a nonnegative integer), substituting it into (5), we have

$$(-1)^{n-1}=(-1)^{n-2k}+i(V,x^*,0)+\cdots. \tag{6}$$

It is clear from (6) that there must exist at least a $x^*\neq 0$ such that $i(V,x^*,0)\neq 0$. Because $x^*\in V^{-1}(0)$, i.e., $V(x^*)=0$, Theorem 1 holds. Now we come to prove $i(V,0,0)=(-1)^{n-2k}$.

Since $V^{-1}(0)$ is a finite set, 0 is an isolated solution to $V(x)=0$. Let U is such a sufficiently small neighborhood of O that 0 is the unique solution to $V(x)=0$ in U. By the definition of index of zero points, we have

$$i(V,0,0)=d(V,U,0)=\operatorname{sgn}(\det V_x(0))=\operatorname{sgn}(\prod_{i=1}^n\sigma_i),$$

where $V_x(0)$ is Jacobi matrix of $V(x)$ at $x=0$, σ_i is the eigenvalue of $V_x(0)$. The properties of σ_i must be carefully investigated to calculate $i(V,0,0)$.

By paper [2], $V_x(0)=\exp(Jt_0)-I$, where I is the identity matrix. By Lemma 4 [2], $t_0=t(0)=\frac{2\pi}{b}>0$. Hence, $\exp(\mu t_0)-1$ is the eigenvalue of $V_x(0)$ with the same multiplicity as μ if μ is the eigenvalue of J.

Firstly, the eigenvalue $\mu=0$ corresponds the eigenvalue -1 of $V_x(0)$ with multiplicity $2r$, hence $\mu=0$ has not any influence on the value of $\operatorname{sgn}(\prod_{i=1}^n\sigma_i)$.

Secondly, let $\alpha \pm i\beta$ be the imaginary eigenvalue of J, then

$$\exp[(\alpha \pm i\beta)t_0] - 1 = [e^{\alpha t_0}\cos(\beta t_0) - 1] \pm ie^{\alpha t_0}\sin(\beta t_0)$$

are the eigenvalues of $V_x(0)$.

When $\alpha \neq 0, |\exp[(\alpha \pm i\beta)t_0]| = e^{\alpha t_0} \neq 1$, $\exp[(\alpha \pm i\beta)t_0] - 1 \neq 0$ are a pair of conjugate complex eigenvalues of $V_x(0)$ with the same multiplicities.Therefore the imaginary eigenvalues of J with non-zero real parts have not any influence on the value of $\mathrm{sgn}(\prod_{i=1}^{n}\sigma_i)$.

When $\alpha = 0$, $\exp(\pm i\beta)t_0) - 1 = [\cos(\beta t_0) - 1] \pm i\sin(\beta t_0)$. It is known from condition (ii) of Theorem 1 that $\exp(\pm i\beta t_0) - 1 \neq 0$ holds for any integer m and are a pair of conjugate eigenvalue of $V_x(0)$ with the same multiplicities. Hence the pure imaginary eigenvalues of J also have not any influence on the value of $\mathrm{sgn}(\prod_{i=1}^{n}\sigma_i)$.

Thirdly, if μ is a positive eigenvalue of J, then $exp(\mu t_0) - 1$ is also a positive eigenvalue of $V_x(0)$. Hence the positive eigenvalues of J have not any influence on the value of $\mathrm{sgn}(\prod_{i=1}^{n}\sigma_i)$.

Finally, if μ is a negative eigenvalue of J,then $exp(\mu t_0) - 1$ is also the negative eigenvalue of $V_x(0)$. Therefore, the value of $\mathrm{sgn}(\prod_{i=1}^{n}\sigma_i)$ is only determined by the negative eigenvalues of $V_x(0)$. Suppose that matrix J has l negative eigenvalues in which multiple roots are numbered according to their multiplicities, then we have $i(V,0,0) = \mathrm{sgn}(\prod_{i=1}^{n}\sigma_i) = (-1)^l$.

The relation between the eigenvalues of J and $\exp(Jt)$ is referred to paper [5].

Since 0 is an eigenvalue of J with multiplicity $2r$ and pure imaginary roots occur in pairs, the number of eigenvalues with zero real parts is even. Among the eigenvalues of J with non-zero real parts are the $2p$ roots with positive real parts (including positive roots), and the imaginary eigenvalues occur in conjugate pairs, hence J have $n - 2k$ negative eigenvalues where k is a nonnegative integer,i.e.,$l = n - 2k$. In brief, we have $i(V,0,0) = (-1)^{n-2k}$. The proof of Theorem 1 is completed.

4. A parallel result

Assume $f_{2i-1}(x_1, x_2, \cdots, x_{2i-2}, 0, 0, x_{2i+1}, \cdots, x_n) = 0, f_{2i}(x_1, x_2, \cdots, x_{2i-2}, 0, 0, x_{2i+1}, \cdots, x_n) = 0$, where i is a positive integer,$1 \leq i \leq [\frac{n}{2}]$, $[\frac{n}{2}]$ is the maximal integer part of $\frac{n}{2}$.

Define the transformation $\widetilde{C}$: $y \longrightarrow x, y \in R^n$ as follows

$$\begin{aligned} x_j &= y_j \quad 1 \leq j \leq n, \quad j \neq 2i-1, 2i, \\ x_{2i-1} &= y_{2i}\cos y_{2i-1}, \\ x_{2i} &= y_{2i}\sin y_{2i-1}. \end{aligned} \tag{7}$$

Through transformation (7), system (1) or (2) is changed into

$$\dot{y}_j = \widetilde{g_j}\,(y_1, y_2, \cdots, y_n), \quad 1 \leq j \leq n, \tag{8}$$

where $\widetilde{g_j}$ is similarly defined with g_j.

Set $\widetilde{M^C}=\{y \in R^n | \widetilde{C}(y) \in M\}$. After a similar proof with Theorem 1, we obtain

Theorem 2 *Assume that system (1) satisfies*

(i) condition (i) of Theorem 1;

(ii) let $b=[-4f_{2i-1,2i}f_{2i,2i-1}-(f_{2i-1,2i-1}-f_{2i,2i})^2]^{\frac{1}{2}}/2$, *the remaining part is the same with condition (ii) of Theorem 1;*

(iv) system (1) or (2) can be changed into (8) by transformation (7), where $\widetilde{g}_j \in C^1(\widetilde{M^C})$ *and* $\widetilde{g}_{2i-1} \neq 0$ *on* $\widetilde{M^C}$,

then there exists at least a nontrivial periodic orbit to system (1) in set M.

Remark The systematic and perfect results have been obtained on the periodic solutions to linear autonomous system under the critical condition. In order to apply Theorem 1 or 2 to a linear system, it must have a positively invariant compact set and satisfies the rotated condition (iv). So the results in this paper are only available to certain kinds of linear systems,but Theorem 1 or 2 are mainly aimed at the critical case of nonlinear autonomous system in higher dimensional space.

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具非双曲型奇点高维自治系统周期轨道的存在性

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摘 要: 本文利用非线性泛函分析拓扑度理论与常微分方程定性分析相结合的方法讨论了奇点为非双曲型时高维自治系统周期轨道的存在性, 得到了系统存在非常数周期轨道的充分条件.