Journal of Mathematical Research & Exposition
Vol. 19, Supp., 183-188, April, 1999

Weighted Approximation by Hight Order Interpolation in L_w^p Space*

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Abstract The paper gives a new way of constructing Hermite-Fejer and Hermite interpolatory polynomials with the nodes of the roots of first kind of Chebyshev polynomials and gives the approximation order of these two kinds of operators. The approximation orders are described with the best rate of approximation of f by polynomials of degree $\le N = (q+1)n-1$ in L_w^p spaces.

Keywords: Hermite-Fejer interpolation, Hermite interpolation, approximation order.

Classification: AMS(1991) 41A05, 42C05/CLC O174.42

Document code: A **Article ID**: 1000-341X(1999)Supp-0183-06

1. Introduction

Let $X_n = \{x_k = \cos\theta_k: \theta_k = \frac{(2k-1)\pi}{2n}\}$ be the first kind Chebyshev polynomial of n-th degree, q be nonnegative integer and $w(x) = (1-x^2)^{-\frac{1}{2}}$.

For a given f defined on $[-1, 1]$, it is well-known that there exists a unique polynomial $H_N(f, x)$ of degree at most $N = (q+1) - 1$ which satisfies the following conditions

$$H_N(f, x_k) = f(x_k), \quad 1 \le k \le n, \tag{1.1}$$

$$H_N^{(j)}(f, x_k) = 0, \quad 1 \le k \le n,\ 1 \le j \le q, \tag{1.2}$$

and there exists a unique polynomial $H_N^*(f, x)$ of degree at most $N = (q+1)n - q$ which satisfies the condition

$$H_N^{*(j)}(f, x_k) = f^{(j)}(x_k) \quad 1 \le k \le n,\ 1 \le j \le q. \tag{1.3}$$

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It was proved by R. Sakai[1] and P. Vertesi[2] that $H_N(f,x)$ convergences to $f(x)$ uniformly on $[-1,1]$ for $f \in C[-1,1]$, whenever the fixed q is odd, and

$$\left|f(x) - H_n(f,x)\right| = O\left(\frac{1}{n}\right)\sum_{k=1}^{n}\omega\left(f, \frac{\sqrt{1-x^2}}{k} + \frac{1}{k^2}\right), \quad x \in [-1,1]. \tag{1.4}$$

Denoted by $L_w^p\ (0 < p < +\infty)$ the set of all functions f such that $\|f\|_{pw} = \left(\int_{-1}^{1}|f|^p w(x)\,dx\right) < +\infty$ and by $C^r[-1,1]$ the set of the r-times continuously differentiable function on $[-1,1]$.

An interesting problem is to considers the mean norm approximation in $C[-1,1]$ spaces. In [3] Wang and Shen proved that

$$\|H_N(f) - f\|_{pw} = O(1)\,\omega\left(f, \frac{1}{n}\right), \quad f \in C[-1,1]\ (0 < p < +\infty), \tag{1.5}$$

where $\omega(f,t) = \sup_{|u_1-u_2|\le t}\max_{-1\le x\le 1}\left|f(x+u_1) - f(x+u_2)\right|$.

When $q = 1$, the same result was given by A. K. Varma and J. Prasad (see [4]).

Let $E_n(f)$ be the best rate of approximation of f by polynomials of degree $\le n$ in the sup-norm on $[-1,1]$.

For the operator $H_N^*(f)$, A. K. Varma and J. Prasad (see [4]) proved (with $q = 1$)

$$\|H^*_{2n-1}(f) - f\|_{pw} = O(1)n^{-1}E_{2n-2}(f'), \quad f \in C^1[-1,1]. \tag{1.6}$$

Recently Min Guohua (see [5]) proved

$$\left(\int_{-1}^{1}\sqrt{1-x^2}\left|H^{*\prime}_{2n-1}(f,x) - f'(x)\right|^2 dx\right)^{\frac{1}{2}} \le CE_{2n-1}(f'). \tag{1.7}$$

The generality result about operators $H_N^*(f,x)$ were given by Wang and Shen[3]

$$\|H_N^*(f) - f\|_{pw} = O(1)n^{-q}E_{N-q}(f^{(q)}), \quad f \in C^q[-1,1] \tag{1.8}$$

The most interesting problems are considering the degree of approximation of operators $H_N(f,x)$ and $H_N^*(f,x)$ in L_w^p spaces. In [3], Wang and Sheng considered the interpolation problem:

$$K_N(f,x_k) = \frac{n}{\pi}\int_{\theta_{k-1}}^{\theta_k} f(\cos\theta)\,d\theta, \quad 1 \le k \le n, \tag{1.9}$$

$$K_N^{(j)}(f,x_k) = 0, \quad 1 \le j \le q,$$

and proved following

Theorem A *Let $f \in L_w^p\ (1 < p < \infty)$, q be a fixed non-negative integer. Then*

$$\|K_N(f) - f\|_{pw} = O(1)\,\omega\left(f, \frac{1}{n}\right)_p,$$

where $\omega(f,t)_p$ denote the L_w^p modulus of smoothness of f.

$$\omega(f,\delta)_p = \sup_{|u_1-u_2|\le\delta}\left(\int_{0}^{2\pi}\left|f(\cos(\theta+u_1)) - f(\cos(\theta+u_2))\right|^p d\theta\right)^{\frac{1}{p}}.$$

Let $E_n(f)_{pw} = \inf_{P\in\pi_n}\|f - p_n\|_{pw}$ be the best rate of approximation of f by polynomials of degree $\le n$ in the norm $\|\ \ \|_{pw}$. Hence, it is natural to ask

(i) There exists the same results about global approximation as (1.4) in $\|\cdot\|_{pw}$ norm?

(ii) Could we describe the degree of convergence of operators $H_N(f,x)$ and $H_N^*(f,x)$ in L_w^p spaces with $E_n(f)_{pw}$?

For this purpose we shall modify the operators $H_N(f,x)$ and $H_N^*(f,x)$ as following:

Let $T_0(x) = \frac{1}{\sqrt{\pi}}$, $T_k(x) = \frac{2}{\sqrt{\pi}}\cos k\arccos x$, $k = 1, 2, 3, \cdots$ are Chebyshev polynomial, which are known as an orthogonal system with weight $w(x)$ on $[-1, 1]$. Let

$$f(x) \sim \sum_{k=1}^{n} c_k T_k(x), \quad x \in [-1, 1] \tag{1.10}$$

be the Chebyshev-Fourier series of $f(x)$. Where $c_k = \int_{-1}^{1} f(x) T_k(x) w(x)\,dx$.

Let $S_n(f,x) = \sum_{k=1}^{n} c_k T_k(x)$ be the partial sum of (1.10), then by a well known results (see [6]) we known $\|S_n(f)\|_{pw} \le C\|f\|_{pw}$, where C is a constant independent of n.

Let

$$\sigma_N(f,x) = \frac{1}{N}\sum_{k=N}^{2N-1} S_k(f,x), \quad N = (q+1)n - 1 \tag{1.11}$$

be the Vallee-Poussin sums of the series (1.10), and the following modufied interplation:

$$S_N^*(f,x_k) = \sigma_N(f,x_k), \quad k = 1, 2, 3, \cdots, n.$$

$$S_N^{*(j)}(f,x_k) = 0, \quad k = 1, 2, \cdots, n, \ 1 \le j \le q, \tag{1.12}$$

and

$$B_N^{(j)}(f,x_k) = \sigma_N^{(j)}(f,x_k) \quad 1 \le j \le q. \tag{1.13}$$

Then we have

Theroem *Let $f \in L_w^p$ $(1 < p < +\infty)$, then there exists constant $C > 0$ such that*

$$\|S_N^*(f) - f\|_{pw} \le \frac{C}{n}\sum_{k=0}^{N} E_k(f)_{pw}, \tag{1.14}$$

and

$$\|B_N(f) - f\|_{pw} \le C E_N(f)_{pw}. \tag{1.15}$$

In this paper, the C always stands for constants independent of n, it may be different in different places, and $A = O(1)$ means A is bounded.

2 Some Lemmas

Lemma 1[3], [7] *Let $q \ge 0$ and $R_N(x)$ be a polynomial of degree at most $N = (q+1)n - 1$. Then*

$$\left(\int_{-1}^{1} |R_N(x)|^p w(x)\,dx\right)^{\frac{1}{p}} \le C\left(\sum_{k=1}^{n}\sum_{j=0}^{q} \frac{|(\sqrt{1-x_k^2})^j R_N^{(j)}(x_k)|^p}{n^{1+jp}}\right)^{\frac{1}{p}}. \tag{2.1}$$

Lemma 2[8] *For every $0 < p < \infty$ and Jacobi weight $u(x)$. Then for every $c > 0$ and polynomial $R(x)$ of degree at most cn, we have*

$$\frac{1}{n}\sum_{k=1}^{n}|R(x_k)|^p u(x_k) = O(1)\int_{-1}^{1}|R(t)|^p u(t)w(t)\mathrm{d}t. \tag{2.2}$$

Lemma 3[3],[9],[10] *Let $p_n(x)$ be a polynomial of degree $\le n-1$, $r = 1, 2, \cdots$. Then for $1 \le p < \infty$ we have*

$$\|(\sqrt{1-x^2})^j p_n^{(j)}(x)\|_{pw} = O(1)n^{j-1}\|\sqrt{1-x^2}p_n'(x)\|_{pw}, \tag{2.3}$$

and

$$\|(\sqrt{1-x^2})^j p_n^{(j)}(x)\|_{pw} = O(1)n^j\|p_n(x)\|_{pw}, \tag{2.4}$$

where j are positive integer.

Lemma 4[9] *For p_n satisfying $\|f - p_n\|_{pw} = E_n(f)_{pw}$ we have*

$$\|\varphi(x)p_n^{(r)}(x)\|_{pw} \le Cn^r\int_0^{\frac{1}{n}}\frac{\Omega_\varphi^r(f,\tau)_{pw}}{\tau}\mathrm{d}\tau, \tag{2.5}$$

$$\|\varphi(x)p_n^{(r)}(x)\|_{pw} \le C\sum_{k=1}^{n}(k+1)^{r-1}E_k(f)_{pw}, \tag{2.6}$$

where $\varphi(x) = \sqrt{1-x^2}$ and

$$\Omega_\varphi^r(f,\tau)_{pw} = \sup_{0<h\le t}\|\Delta_{h\varphi}^r f\|_{pw[-1+2r^2h^2,1-2r^2h^2]}, \quad \Delta_{h\varphi}^r f = \Delta_{h\varphi}(\Delta_{h\varphi}^{r-1}f),$$

and $\Delta_{h\varphi}f(x) = f\left(x + \frac{h}{2}\varphi(x)\right) - f\left(x - \frac{h}{2}\varphi(x)\right)$.

3. The proof of Theorem

The proof of (1.14): Let $p_N(x)$ satisfy $\|f - p_N\|_{pw} = E_N(f)_{pw}$, then by Lemma 1 and the interplating conditions, we have

$$\begin{aligned}\|S_N^*(f) - f\|_{pw} &\le \|S_N^*(f - p_N)\|_{pw} + \|S_N^*(p_N) - p_N\|_{pw} + \|p_N - f\|_{pw} \\ &= A + B + C,\end{aligned}$$

$$\begin{aligned}A &= \|S_N^*(f - p_N)\|_{pw} \le C\left(\sum_{k=1}^{n}\sum_{j=0}^{q}\frac{|(\sqrt{1-x_k^2})^j S_N^*(f - p_N, x_k)|^p}{n^{1+jp}}\right)^{\frac{1}{p}} \\ &= C\left(\sum_{k=1}^{n}\frac{|\sigma_N(f - p_N, x_k)|^p}{n}\right)^{\frac{1}{p}} \le C\left(\int_{-1}^{1}|\sigma_N(f - p_N, x)|^p w(x)\mathrm{d}x\right)^{\frac{1}{p}} \\ &= CE_N(f)_{pw}.\end{aligned}$$

Since $\sigma(p_N, x) = p_N(x)$ we have

$$B = \|S_N^*(p_N) - p_N\|_{pw} \le C\Big(\sum_{k=1}^{n}\sum_{j=0}^{q}\frac{|(\sqrt{1-x_k^2})^j(S_N^{*(j)}(p_N, x_k) - p_N^{(j)}(x_k))|^p}{n^{1+jp}}\Big)^{\frac{1}{p}}$$

$$= C\Big(\sum_{k=1}^{n}\frac{|S_N^*(p_N, x_k) - p_N(x_k)|^p}{n} + \sum_{k=1}^{n}\sum_{j=1}^{q}\frac{|(\sqrt{1-x_k^2})^j p_N^{(j)}(x_k)|^p}{n^{1+jp}}\Big)^{\frac{1}{p}}$$

$$= C\Big(\sum_{k=1}^{n}\frac{|\sigma_N(p_N, x_k) - p_N(x_k)|^p}{n} + \sum_{k=1}^{n}\sum_{j=1}^{q}\frac{|(\sqrt{1-x_k^2})^j p_N^{(j)}(x_k)|^p}{n^{1+jp}}\Big)^{\frac{1}{p}}$$

$$\le C\Big(\|\sigma_N(p_N) - p_N\|_{pw} + \sum_{j=1}^{q}\frac{\|(\sqrt{1-x_k^2})^j p_N^{(j)}(x)\|_{pw}}{n^j}\Big)$$

$$\le C\frac{1}{n}\|\sqrt{1-x^2}\,p_N'(x)\|_{pw} \le \frac{C}{n}\sum_{k=1}^{N}E_k(f)_{pw}$$

$$D = E_N(f)_{pw}.$$

The (1. 4) be proved.

The proof of (1. 15): By Lemma 1, Lemma 2 and the interpolating condition that

$$\|B_N(f) - f\|_{pw} \le \|B_N(f - p_N)\|_{pw} + \|f - p_N\|_{pw}$$

$$\le C\Big(\sum_{k=1}^{n}\sum_{j=0}^{q}\frac{|(\sqrt{1-x_k^2})^j B_N^{(j)}(f - p_N, x_k)|^p}{n^{1+jp}}\Big)^{\frac{1}{p}} + E_N(f)_{pw}$$

$$\le C\Big(\sum_{k=1}^{n}\sum_{j=0}^{q}\frac{|(\sqrt{1-x_k^2})^j \sigma_N^{(j)}(f - p_N, x_k)|^p}{n^{1+jp}}\Big)^{\frac{1}{p}} + E_N(f)_{pw}$$

$$\le C\Big(\sum_{j=0}^{q}\frac{|(\sqrt{1-x^2})^j \sigma_N^{(j)}(f - p_N)|^p}{n^{jp}}\Big)^{\frac{1}{p}} + E_N(f)_{pw}$$

$$\le C(\|\sigma_N(f - p_N)\|_{pw} + E_N(f)_{pw})$$

Because of the boundedness of $S_n(f, x)$ in L_w^p spaces, we can prove that

$$\|\sigma_N(f - p_N)\|_{pw} \le C\|f - p_N\|_{pw} = CE_N(f)_{pw}.$$

Where we have used the fact that $\sigma_N(p_N, x) = p_N(x)$, $B_N(p_N, x) = p_N(x)$ and Lemma 2.

Corollary 1 *For $f(x) \in L_w^p (1 < p < +\infty)$, we have*

$$\|S_N^*(f) - f\|_{pw} \le \frac{C}{n}\sum_{k=0}^{N}\int_0^{\frac{1}{k}}\frac{\Omega_\varphi^r(f, \tau)_{pw}}{\tau}d\tau,\quad \|B_N(f) - f\|_{pw} \le C\int_0^{\frac{1}{N}}\frac{\Omega_\varphi^r(f, \tau)_{pw}}{\tau}d\tau$$

Proof It can be proved by the fact that $E_n(f)_{pw} \le C\int_0^{\frac{1}{n}}\frac{\Omega_\varphi^r(f, \tau)_{pw}}{\tau}d\tau$

Corollary 2 *For* $f(x) \in L_w^p(1 < p < +\infty)$ *we have*

$$\left(\int_{-1}^{1} \sqrt{1 - x^2}\,\left|B_N(f, x) - f(x)\right|^p dx\right)^{\frac{1}{p}} \leq CE_N(f)_{pw}$$

Proof With the help of the fact that $\sqrt{1 - x^2} \leq \frac{1}{\sqrt{1 - x^2}}$ the Corollary 2 is a corollary of (1.15).

Therefore, it can be proved with Markov-inequality that (1.7) is a corollary of (1.6) with $p = 2$ and the results of the theorem are also tenable for some Jacobi weight function.

References

[1] Sakai R. *Hermite-Fejer interpolation prescribing higher order derivatives* [C]. Progress in Approximation Theory (P. Nevai and Pinkus, eds), Academic Press, New York, 1991, 731-759.

[2] Vertesi P. *Hermite-Fejer interpolation of higher I* [J]. Acta Math. Hung., 1989, 54(1-2): 135-152.

[3] Wang Ziyu and Shen Xiechang. *Weighted* L^p *approximation by the modified higher order Hermite-Fejer interpolation* [J]. Advances in Math., 1994, **23**(4): 342-353.

[4] Varma A K and Prasad J. *An analogue of a problem of* L_p *convergence of interpolatory process* [J]. J. Approx. Theory, 1989, **56**: 225-240.

[5] Min Guohua. *On mean convergence of the derivatives of Hermite interpolation operator* [J]. Advances in Mathematices, 1990, **21**(3): 329-333.

[6] Yuan Xu. *Mean convergence of generalized Jacobi series and interpolation polynomials* [J]. J. of Approx., 1993, **72**: 237-251.

[7] Yuan Xu. *The Marcinkiewicz-Zygmund inequality with derivatives* [J]. Approx. Theory and Its Appl, 1991, **7**(1): 100-107.

[8] Nevai P. *Orthogonal polynomials* [J]. Mem. Amer. Math. Soc., 1979, 213.

[9] Ditzian Z and Totik V. *Moduli of Smoothness* [M]. Springer-Verlag, New York Berlin Heidelberg, London Paris Tokyo.

[10] Lubinsky D S and Nevai P. *Markov-Bernstein inequalities revisited* [J]. Approx. Theory and Its Appl, 1987, **3**(4): 98-119.

高阶插值在L_w^p空间中的带权逼近

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摘　要

构造了一类新的Hermite及Hermite-Féjer高阶插值多项式, 研究了其逼近阶, 逼近上界用N阶多项式最佳逼近给出.