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A Necessary and Sufficient Condition on Diffeomorphisms *

Wei-ping LI[1], *YU Yong-xi*[2]

(1. Dept of Math., Oklahoma State University, Stillwater, Oklahoma 74078-0613, U.S.A;
2. Dept. of Math., Suzhou Univeristy, 215006)

Abstract: We introduce anti-sheaves for C^r-manifolds and a category of anti-sheaves. The adjoint equivalence between the category of anti-sheaves and the category of C^r-manifolds is established. Using such an equivalence, we obtain a characterization for deciding whether two given manifolds are C^r-diffeomorphic in terms of inherent $W.(G)$-sheaves. This provides the first known criterion for determining whether two given manifolds are diffeomorphic.

Key words: C^r-dffeomorphism; adjoint equivalence; inherent W.G.-sheaf.

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1. Introduction

It is known that a fundamental task of differential topology is to find methods for deciding whether two given manifolds are C^r-diffeomorphic to each other ([5] p.16) for $r = 0, 1, \cdots, \infty$. Kervarie [6] and Smale [9] found some compact manifolds with dimensions ≥ 8 having no differential structure whatsoever. It is known that such a "nonsmoothable" manifold must have dimension at least 4. In the last decade, Donaldson[1,2] showed that there are lots of compact, simply connected, oriented 4-manifolds having no smooth structure. Gompf[3] showed that there are uncountably many inequivalent differential structures on $\mathbf{R}^4$. The very basic question on differential topology about deciding whether two given manifolds are diffeomorphic remains unanswered.

It is not clear that there exists a differential invariant system which can identify the diffeomorphic C^r-manifolds. All the known smooth invariants are good for determining whether two manifolds are not diffeomorphic. For example, Donaldson invariants and Seiberg-Witten invariants can not be used to tell whether two smooth 4-manifolds are diffeomorphic. Using the anti-sheaf and the inherent W.(G)-sheaf introduced in this paper,

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Biography: Wei-ping LI (1963-), male, born in Deyang city, Sichuan province, PRC. Degree Ph.D from Michigan State University, Research interest: low-dimensional topology and differential (symplectic) geometry.

we have at least a characterization for the C^r-diffeomorphisms. **Thus we obtain a method for deciding whether two given manifolds are diffeomorphic by the anti-sheaves.** *Up to authors' knowledge, this is the first known criterion for determining whether two given manifolds are diffeomorphic.* Like the differential invariants are hard to compute, our criterion (Theorem B) is also not easy to verify. Our characterization via anti-sheaves inherits the diffeomorphic property from the sheaf theory point of view (see [4,8]).

In this paper, we define a notion of the anti-sheaf (§3) and a category of anti-sheaves (§4). By studying the structure of the category of anti-sheaves, we find a necessary and sufficient condition for two manifolds diffeomorphic to each other in terms of anti-sheaves. Therefore we provide a method to distinguish the equivalence of C^r-manifolds. We have proved the following theorems.

Theorem A *The adjunction $\langle AM, \mathbf{U}, \eta, \varepsilon\rangle : \mathcal{AS} \rightharpoonup C^r$ is an adjoint equivalence, where $\mathcal{AS}$ is the category of all anti-sheaves on $\mathbf{R}^n$, $\mathbf{U}$ is a forgetful functor from $\mathcal{AS}$ to C^r (see §5 for η, ε).*

Theorem A shows that the set $\{\phi_j\phi_i^{-1}\}$ is a principal part of a C^r differential structure $\Phi = \{\phi_i | i \in I\}$. By forgetting some structure of C^r (the category of all C^r n-manifolds), we get the category $\mathcal{AS}$ of all n-anti-sheaves on $\mathbf{R}^n$ and their equivalence. Then the necessary and sufficient condition on C^r-diffeomorphisms is proved in Theorem B. An application of this theory has been obtained by the second author in [10].

Theorem B *Two manifolds $(X,\Phi),(Y,\Psi)$ are diffeomorphic to each other, $(X,\Phi) \approx (Y,\Psi)$ if and only if there are a (c)-basis B_F of X and a (c)-basis B_H of Y such that*

$$[G_F, f]_{B_F} \stackrel{m}{=} [G_H, h]_{B_H},$$

for a bijective order preserving map $m : B_F \to B_H$. Here (X,Φ) and (Y,Ψ) are two C^r n-manifolds, $[G_F, f]_{B_F}$ is an inherent $W.(G)$ sheaf of (X,Φ) (see §6).

Notations: Let $r \geq 0$ and $n \geq 1$ be integers, and let $\mathbf{H} = \{x \in \mathbf{R}^n; \lambda(x) \geq 0\}$ be an n-half space ([5] p 29). Let U and V be open subsets of $\mathbf{H}$, $U \approx V$ means that U is C^r-diffeomorphic to V, $A \cong B$ means that A is isomorphic to B in a category. $1_U : U \to U$ is an identity map, $\hookrightarrow$ or i denotes an inclusion map. If $f : U \to V$ is a C^r map and $f^{-1} : f(U) \to U$ is also of class C^r, we write $f : U \rightsquigarrow V$. A formula

$$A \xrightarrow{\alpha} B \xrightarrow{\beta} C = A \xrightarrow{\gamma} D \xrightarrow{\delta} C,$$

means that the following diagram is commutative.

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & B \\ \gamma \downarrow & & \downarrow \beta \\ D & \xrightarrow{\delta} & C. \end{array}$$

This paper is organized as follows. In §2, we describe the basic limit property of differential structures. The definition of an anti-sheaf is given in section 3. §4 is devoted to construct the category of anti-sheaves. Then we prove Theorem A in §5 and Theorem

B in §6. Some related topics are also discussed in these sections.

2. Limit property of differential structures

Let θ_n^r be the category: (1) Objects: all open subsets of $\mathbf{H}$, (2) Arrows: all C^r maps between two open subsets of $\mathbf{H}$.

Definition 2.1 *For a C^r n-manifold (M, Φ) with its differential structure $\Phi = \{(\phi_U, U) | U \in M_\Phi\}$, then M_Φ is called a (c)-basis of M if $V \neq \emptyset$ is an nonempty open subset of U $(U \in M_\Phi)$, then $V \in M_\Phi$.*

It is clear that there always exists a (c)-basis for any C^r n-manifold. We suppose that $\{U_\lambda | \lambda \in \Lambda\} \subset M_\Phi$ and $U = \cup_{\lambda \in \Lambda} U_\lambda$, then $U \in M_\Phi$. Let J be an index category: If $\lambda, \lambda', \lambda''$ and $U_\lambda = U_{\lambda'} \cap U_{\lambda''}$, then we have $\Lambda \subset J$ and complete Λ in the following way. For $\lambda, \lambda' \in \Lambda$, $U_\lambda \cap U_{\lambda'} \neq \emptyset$ and there is no $l \in \Lambda$ such that $U_l = U_\lambda \cap U_{\lambda'}$, we set $j = \lambda\lambda' \in J$ such that $U_j = U_\lambda \cap U_{\lambda'}$.

Let $F \in (\theta_n^r)^J$ be a functor category with objects the functors $J \to \theta_n^r$ and morphisms the natural transformations between two such functors (c.f. [7] p 40) such that

$$\begin{gathered} F_{\lambda\lambda'} = F(\lambda\lambda') = \phi_{\lambda\lambda'}(U_\lambda \cap U_{\lambda'}), \quad F_\lambda = F(\lambda) = \phi_\lambda(U_\lambda), \quad F_{\lambda'} = F(\lambda'); \\ F(\alpha_\lambda) = \phi_\lambda \phi_{\lambda\lambda'}^{-1}, \quad F(\beta_{\lambda'}) = \phi_{\lambda'} \phi_{\lambda\lambda'}^{-1}, \\ F(1_{\lambda\lambda'}) = i_{\phi_{\lambda\lambda'}(U_\lambda \cap U_{\lambda'})}, \quad F(1_\lambda) = i_{\phi_\lambda(U_\lambda)}, \quad F(1_{\lambda'}) = i_{\phi_{\lambda'}(U_{\lambda'})}. \end{gathered} \tag{2.1}$$

Then we have the following proposition.

Proposition 2.2 $\phi(U) = \lim_{\to} F$.

Proof Let $\mathbf{D} : \theta_n^r \to (\theta_n^r)^J$ be a diagonal functor (see [7] p 67). We define $\eta : F \to \mathbf{D}\phi(U)$ as follows: Set $j = \lambda\lambda'$ for $U_j = U_\lambda \cap U_{\lambda'}$ and $\eta_j = \eta_{\lambda\lambda'} = \phi\phi_{\lambda\lambda'}^{-1} : F(j) \to \phi(U)$; if $\lambda \in \Lambda$, $\eta_\lambda = \phi\phi_\lambda^{-1} : F(\lambda) \to \phi(U)$.

Thus $\eta : F \dot{\to} \mathbf{D}\phi(U)$ is a natural transformation for $\eta_\lambda F(\alpha_\lambda) = \eta_j$ by (2.1), so $\eta \in Arr(\theta_n^r)^J$. For each $V \in ob\theta_n^r$, $g : F \dot{\to} \mathbf{D}V \in Arr(\theta_n^r)^J$, define $h : \phi(U) \to V$:

For $t \in \phi(U)$, $h(t) = g_\lambda \phi_\lambda(\phi^{-1}(t))$ if $\phi^{-1}(t) \in U_\lambda$.

We check first that h is well-defined: for $\phi^{-1}(t) \in U_\lambda \cap U_{\lambda'}$,

$$\begin{aligned} g_{\lambda'}(\phi_{\lambda'}(\phi^{-1}(t))) &= g_{\lambda'}(\phi_\lambda(\phi_{\lambda\lambda'}^{-1}(\phi^{-1}(t)))) = g_{\lambda\lambda'}(\phi_{\lambda\lambda'}^{-1}(\phi^{-1}(t))) \\ &= g_\lambda((\phi_\lambda\phi_{\lambda\lambda'})(\phi_{\lambda\lambda'}^{-1}(\phi^{-1}(t))) = g_\lambda(\phi_\lambda(\phi^{-1}(t))). \end{aligned}$$

$h \in C^r$, so $h \in Arr\theta_n^r$. We are going to show that $\mathbf{D}h \cdot \eta = g$. For $j = \lambda\lambda'$ and for each $t \in \phi_{\lambda\lambda'}(U_\lambda \cap U_{\lambda'}) = F_{\lambda\lambda'}$, we have

$$\begin{aligned} h_{\lambda\lambda'}(\eta_{\lambda\lambda'}(t)) &= h_{\lambda\lambda'}(\phi\phi_{\lambda\lambda'}^{-1}(t)) \\ &= g_{\lambda\lambda'}\phi_{\lambda\lambda'}(\phi^{-1}(\phi\phi_{\lambda\lambda'}^{-1}(t))) \quad \text{(by definition of } h) \\ &= g_{\lambda\lambda'}(t). \end{aligned}$$

Such a map h is unique (see also the proof of Proposition 3.4), so $\phi(U) = \lim_{\to} F$ ([7] p 67). □

3. Axioms of Anti-sheaves

Definition 3.1 *Let T be a (c)-basis. A C^r n anti-presheaf on T is characterized by the following:*

AP1. Each $U \in T$ is assigned a nonempty set S_U of symbols and a nonempty set R_U of open subsets of **H** *with the following property. For $O \in R_U$, if an open set $O' \approx O$ then $O' \in R_U$ and there is a bijective map $\alpha_U : S_U \to R_U$. For $p \in S_U$ and $\alpha_U(p) = O$, we denote $p(U)$ for O.*

AP2. Given $U, V \in T$, if $V \subseteq U$ then for all $p'(U)$ and all $p(V)$ there exists a unique $\rho^{p(V)}_{p'(U)} : p(V) \rightsquigarrow p'(U)$ (recall $\rightsquigarrow$ a C^r-diffeomorphism) such that

(i) $\rho^{p(V)}_{p'(U)}$ is a C^r-diffeomorphism and $\rho^{p(U)}_{p'(U)} = 1_{p(U)}$ if and only if $p = p'$. (ia) If $U_1 \subseteq U$, then $\rho^{p(U_1)}_{p'(U)}$ is an inclusion map for $p \in S_U \cap S_{U_1}$; (ib) if $p(U_1) = p(U)$ then $U_1 = U$.

(ii) If $W \subseteq V \subseteq U$, then for any $p(W), p'(V)$ and $p''(U)$,

$$\rho^{p'(V)}_{p''(U)} \cdot \rho^{p(W)}_{p'(V)} = \rho^{p(W)}_{p''(U)}. \tag{3.1}$$

(iii) If $V \subseteq U$, then for any $p \in S_U$, there exists a unique $q_p \in S_V$ such that $q_p(V) \subseteq p(U)$ and $\rho^{q_p(V)}_{p(U)}$ is an inclusion map.

Remark The difference between an anti-presheaf and a presheaf is to use the C^r-diffeomorphisms in AP2 instead of restrictions. The "anti" terminology comes from AP2 (ii) by comparing with the usual presheaf definition (c.f. [4] Chapter 0 §3 and §5 and [8]).

Definition 3.2 *A C^r n anti-presheaf on T is called a C^r n anti-sheaf if the following holds.*

AS1. For any open subset $O \subseteq p(U)$, there exists a unique open subset $\tilde{U} \subseteq U$ and a unique $q_p \in S_{\tilde{U}}$ such that $q_p(\tilde{U}) = O$ and $\rho^{q_p(\tilde{U})}_{p(U)}$ is an inclusion map.

AS2. If $U, U_i \in T (i \in I)$ and $U = \cup_{i \in I} U_i$, then $p(U) = \cup_{i \in I} p(U_i)$. If $U', V \subseteq U$, then $U' \cap V \neq \emptyset$ if and only if $p(U') \cap p(V) \subseteq p(U' \cap V)$.

AS3. If $\tilde{U} \subseteq U$ then $\rho^{p(U)}_{p'(U)}|_{p(\tilde{U})} = \rho^{p(\tilde{U})}_{p'(\tilde{U})}$.

AS4. If $U, U_i \in T (i \in I)$ and $U = \cup_{i \in I} U_i$, and for each $i \in I$, $p(U_i) = p'(U_i)$ and $p = p'$ for each U_i, then $p = p'$ for U.

Notation The unique q_p is always written as p, then q_p in AP1 (iii) can be written as p. Such a C^r n anti-sheaf is denoted by $\langle p, \rho \rangle_T$.

Proposition 3.3 *For a C^r n anti-sheaf, if $U \subseteq V \subseteq W$, $p(U) \overset{\rho}{\hookrightarrow} p(W)$ and $p(V) \overset{\rho}{\hookrightarrow} p(W)$, then $p(U) \overset{\rho}{\hookrightarrow} p(V)$.*

Proof We have that $\rho^{p(U)}_{p(V)}$ makes sense by AP2 and $\rho^{p(V)}_{p(W)} \rho^{p(U)}_{p(V)} = \rho^{p(U)}_{p(W)}$ by AP2 (ii). So for $x \in p(U)$,

$$x = \rho^{p(U)}_{p(W)}(x) = \rho^{p(V)}_{p(W)}(\rho^{p(U)}_{p(V)}(x)) = \rho^{p(U)}_{p(V)}(x).$$

We define $\eta : F \to \mathbf{D}p(U)$ as follows:

$$\eta_j = \begin{cases} \rho^{\tilde{p_0}(U_j)}_{p(U)} & \text{if } j = \lambda\lambda' \\ \rho^{\tilde{p_\lambda}(U_\lambda)}_{p(U)} & \text{if } j = \lambda \in \Lambda \end{cases} \tag{3.2}$$

Here $F \in (\theta^r_n)^J$, $F(j) = \tilde{p_0}(U_j)$ for $U_j = U_\lambda \cap U_{\lambda'}$ as an element in $S(U_j)$,

$$F(\lambda) = \tilde{p_\lambda}(U_\lambda), \quad F(\lambda') = \tilde{p_{\lambda'}}(U_{\lambda'}), \tag{3.3}$$

$$F(\alpha_\lambda) = \rho^{\tilde{p_0}(U_j)}_{\tilde{p_\lambda}(U_\lambda)}, \quad F(\beta_{\lambda'}) = \rho^{\tilde{p_0}(U_j)}_{\tilde{p_{\lambda'}}(U_{\lambda'})}. \tag{3.4}$$

By (3.2) and (3.4), it is easy to check that

$$\eta_\lambda F(\alpha_\lambda) = \rho^{\tilde{p_\lambda}(U_\lambda)}_{p(U)} \rho^{\tilde{p_0}(U_j)}_{p(U)} = \eta_j. \tag{3.5}$$

So $\eta = (\eta_j)_{j\in J} : F \dot{\to} \mathbf{D}(p(U))$ is a natural transformation.

Proposition 3.4 *If $U, U_\lambda \in T(\lambda \in \Lambda)$ and $U = \cup_{\lambda\in\Lambda} U_\lambda$, then for any set $\{\tilde{p}_\lambda(U_\lambda) | \lambda \in \Lambda\}$*

$$p(U) = \varinjlim F.$$

Proof Let $V \in ob(\theta^r_n)$ be an object and $g : F \dot{\to} \mathbf{D}V \in Arr(\theta^r_n)^J$. We define a map

$$h : p(U) \to V, \quad t \mapsto g_\lambda(\rho^{p(U_\lambda)}_{\tilde{p_\lambda}(U_\lambda)}(t)) \quad \text{for } t \in p(U_\lambda). \tag{3.6}$$

Check that h is well-defined. In fact for any $t \in p(U)$, there exists a $\lambda \in \Lambda$ such that $t \in p(U_\lambda)$ by AS2. Moreover if $t \in p(U_{\lambda'})$ then $U_\lambda \cap U_{\lambda'} \neq \emptyset$. So $j = \lambda\lambda' \in J$ and $t \in p(U_j)$. By AS1, we have

$$\begin{aligned}
g_\lambda(\rho^{p(U_\lambda)}_{\tilde{p_\lambda}(U_\lambda)}(t)) &= g_\lambda(\rho^{p(U_\lambda)}_{\tilde{p_\lambda}(U_\lambda)}(\rho^{p(U_j)}_{p(U_\lambda)}(t))) && \text{(AS1)} \\
&= g_\lambda(\rho^{p(U_j)}_{\tilde{p_\lambda}(U_\lambda)}(t)) && \text{(AP2(ii))} \\
&= g_\lambda(\rho^{\tilde{p_0}(U_j)}_{\tilde{p_\lambda}(U_\lambda)} \rho^{p(U_j)}_{\tilde{p_0}(U_j)}(t)) && (t \in p(U_j)) \\
&= g_\lambda F(\alpha_\lambda)(\rho^{p(U_j)}_{\tilde{p_0}(U_j)}(t)) && (3.4) \\
&= g_j(\rho^{p(U_j)}_{\tilde{p_0}(U_j)}(t)) && (\text{Definition of } g) \\
&= g_{\lambda'} F(\beta_{\lambda'})(\rho^{p(U_j)}_{\tilde{p_0}(U_j)}(t)) && \text{(naturality)} \\
&= g_{\lambda'} \rho^{\tilde{p_0}(U_j)}_{\tilde{p_{\lambda'}}(U_{\lambda'})}(\rho^{p(U_j)}_{\tilde{p_0}(U_j)}(t)) && (3.4) \\
&= g_{\lambda'}(\rho^{p(U_j)}_{\tilde{p_{\lambda'}}(U_{\lambda'})}(t)) && \text{(AP2(ii))} \\
&= g_{\lambda'}(\rho^{p(U_{\lambda'})}_{\tilde{p_{\lambda'}}(U_{\lambda'})}(t)). && \text{(AS1)}
\end{aligned}$$

Next we need to show that $\mathbf{D}h \cdot \eta = g$. For $i \in J$ and $t \in F_i$, we have

$$\eta_i(t) = \rho^{\tilde{p_i}(U_i)}_{p(U)}(t) = \rho^{p(U_i)}_{p(U)}(\rho^{\tilde{p_i}(U_i)}_{p(U_i)}(t)). \tag{3.7}$$

Thus $\eta_i(t) \in p(U_i)$ for the inclusion $\rho_{p(U)}^{p(U_i)}$. Hence

$$\begin{aligned} h\eta_i(t) &= g_i(\rho_{\tilde{p_i}(U_i)}^{p(U_i)}(\eta_i(t))) && (3.6) \\ &= g_i(\rho_{\tilde{p_i}(U_i)}^{p(U_i)}(\rho_{p(U)}^{\tilde{p_i}(U_i)}(t))) && (3.7) \\ &= g_i(t). \end{aligned}$$

Thus we obtain the identity $\mathbf{D}h \cdot \eta = g$. Such a h is unique. If $\mathbf{D}h' \cdot \eta = g$ for h', then by similar arguments used before, we have

$$\begin{aligned} h'(t) &= h'(\rho_{p(U_\lambda)}^{p(U_\lambda)}(t)) == h'(\rho_{p(U_\lambda)}^{\tilde{p_\lambda}(U_\lambda)}\rho_{\tilde{p_\lambda}(U_\lambda)}^{p(U_\lambda)}(t)) \\ &= h'\eta_\lambda(\rho_{\tilde{p_\lambda}(U_\lambda)}^{p(U_\lambda)}(t)) = g_\lambda(\rho_{\tilde{p_\lambda}(U_\lambda)}^{p(U_\lambda)}(t)) = h(t). \end{aligned}$$

The result follows. □

Remark The property $\lim_{\rightarrow} F = p(U)$ corresponds to the axioms (M) and (G) in Proposition 1.7 of [8] §2.1.

4. Category of anti-sheaves

4.1. Quasi-morphism and morphism Let $\langle F, \rho\rangle_T$ and $\langle G, \tau\rangle_W$ be C^r n anti-sheaves. We set the following axioms.

QM1. There exists $B_\alpha = \{U_i\}_{i\in I}$ (a (c)-sub-basis of T), and $W' = \{V_s\}_{s\in\Sigma}$ a subset of W such that for each $U_i \in B_\alpha$ there exists a corresponding open set $V_i \in W'$ such that if $V_i \subseteq \tilde{V_i} \subseteq W'$, then there exist

$$\alpha_{G(\tilde{V_i})}^{F(U_i)} : F(U_i) \to G(\tilde{V_i}) \in C^r,$$

for all $F(U_i)$ and $G(\tilde{V_i})$. Denote by $\Gamma_1 = \{\alpha_{G(\tilde{V_i})}^{F(U_i)} | U_i \in B_\alpha\}$.

QM2. If $\mathrm{Im}(\alpha_{G(V_i)}^{F(U_i)}) \subseteq G(V_i')$ and $V_i \subseteq V_i'$, then there exists

$$\alpha_{G(V_i')}^{F(U_i)} : F(U_i) \to G(V_i') \in C^r,$$

for all $F(U_i)$ and $G(V_i')$. Denote by $\Gamma_2 = \{$all $\alpha_{G(V_i')}^{F(U_i)}$ and all their restrictions $\alpha_{G(V_i')}^{F(U_i)}|_{F(U_i')}\}$. Let $\alpha = \Gamma_1 \cup \Gamma_2$.

QM3. If $U_i, U_j \in B_\alpha$ and $U_i \cap U_j \neq \emptyset$, then $V_i \cap V_j \neq \emptyset$ is an open subset corresponding to $U_i \cap U_j$ and the following diagram is commutative.

$$\begin{array}{ccc} F(U_i \cap U_j) & \overset{\alpha_{G(V_i\cap V_j)}^{F(U_i\cap U_j)}}{\longrightarrow} & G(V_i \cap V_j) \\ \rho \downarrow & & \downarrow \tau \\ F'(U_j) & \overset{\alpha_{G'(V_j)}^{F'(U_j)}}{\longrightarrow} & G'(V_j). \end{array} \qquad (4.1)$$

Definition 4.1 *If the axioms QM1, QM2, and QM3 hold for* $\langle F,\rho\rangle_T$ *and* $\langle G,\tau\rangle_W$*, then* $\alpha : \langle F,\rho\rangle_T \cdots \rightarrow \langle G,\tau\rangle_W$ *is called a* C^r *quasi-morphism on* B_α.

Notation Note that each open set corresponding to $U_i \in B_\alpha$ in QM1 is denoted by a letter with its index i, it is not unique in general.

Definition 4.2 *Let* α *and* β *be quasi-morphisms. If we have*

$$\alpha = \{\alpha_{G(V_i)}^{F(U_i)} | i \in I\} \subseteq \beta = \{\beta_{G(V_s)}^{F(U_s)} | s \in \Sigma\}, \tag{4.2}$$

then α *is called a reduction of* β.

We define a binary relation R in the set of all quasi-morphisms from an anti-sheaf to another one as follows:

$$(\alpha,\beta) \in R \text{ if and only if } \alpha \text{ and } \beta \text{ have their common reduction.} \tag{4.3}$$

Proposition 4.3 *Two reductions of a quasi-morphism have a common reduction.*

Proof Let $\{\alpha_{G(V_i)}^{F(U_i)} | i \in I\}$ and $\beta = \{\beta_{G(V_s)}^{F(U_s)} | s \in \Sigma\}$ be two reductions of $\gamma = \{\gamma_{G(V_w)}^{F(U_w)} | w \in W\}$. Set $A = \{U_i \cap U_s | U_i \cap U_s \neq \emptyset\}$. Then A is a (c) sub-basis of T. We define a quasi-morphism δ on A as follows:

Since $U_i \cap U_s = U_{i'} \subseteq U_i$ and $U_i \cap U_s = U_{s'} \subseteq U_s$, there exist $V_{i'}^s \subseteq V_i$ and $V_{s'}^i \subseteq V_s$ such that $\alpha_{G(V_{i'}^s)}^{F(U_{i'})}$ and $\beta_{G(V_{s'}^i)}^{F(U_{s'})}$ make sense by QM3. Such $V_{i'}^s, V_{s'}^i$ are called *admissible*. So $\gamma_{G(V_{i'}^s)}^{F(V_{i'})}$ and $\gamma_{G(V_{s'}^i)}^{F(U_{s'})}$ are also well-defined. Since $U_{s'} \cap U_{i'} \neq \emptyset$, thus $V_{i'}^s \cap V_{s'}^i \neq \emptyset$ and $\gamma_{G(V_{i'}^s \cap V_{s'}^i)}^{F(U_{i'} \cap U_{s'})}$ is well-defined by QM3. Denote $V_{i'}^s$ by V_i^s and $V_{s'}^i$ by V_s^i. Define

$$\delta = \{\delta_{G(V_i^s \cap V_s^i)}^{F(U_i \cap U_s)} = \gamma_{G(V_i^s \cap V_s^i)}^{F(U_i \cap U_s)} | U_i \cap U_s \in A \text{ and for all admissible } V_i^s \text{ and } V_s^i\}. \tag{4.4}$$

Lemma 4.4 $\delta \subseteq \alpha \cap \beta$.

Proof By QM3,

$$i_{G(V_i^s)}^{G(V_i^s \cap V_s^i)} \gamma_{G(V_i^s \cap V_s^i)}^{F(U_i \cap U_s)} = \gamma_{G(V_i^s)}^{F(U_i \cap U_s)},$$

$$\gamma_{G(V_i^s)}^{F(U_i \cap U_s)} = \alpha_{G(V_i^s)}^{F(U_i')}.$$

$\alpha_{G(V_i^s \cap V_s^i)}^{F(U_i')}$ is well-defined for $Im(\alpha_{G(V_i^s)}^{F(U_i')}) \subseteq G(V_i^s \cap V_s^i)$ by QM2. Since $i_{G(V_i^s)}^{G(V_i^s \cap V_s^i)} \alpha_{G(V_i^s)}^{F(U_i')} = \alpha_{G(V_i^s)}^{F(U_i')}$, so $\gamma_{G(V_i^s \cap V_s^i)}^{F(U_i \cap U_s)} = \alpha_{G(V_i^s \cap V_s^i)}^{F(U_i \cap U_s)}$ and therefore $\delta \subseteq \alpha$. Similarly, $\delta \subseteq \beta$. □

Lemma 4.5 δ *is a quasi-morphism on* A.

Proof It is easy to see that QM1 holds for δ.

QM2: If $Im(\delta^{F(U_i\cap U_s)}_{G(V_i^s\cap V_s^i)})\subseteq G(V_0)\subseteq G(V_i^s\cap V_s^i)$, we have

$$\delta^{F(U_i\cap U_s)}_{G(V_i^s\cap V_s^i)}=\alpha^{F(U_i\cap U_s)}_{G(V_i^s\cap V_s^i)}=\beta^{F(U_i\cap U_s)}_{G(V_i^s\cap V_s^i)}, \tag{4.5}$$

so that $\alpha^{F(U_i\cap U_s)}_{G(V_0)}$ and $\beta^{F(U_i\cap U_s)}_{G(V_0)}$ exist. Let $U_i\cap U_s=U_{i'}=U_{s'}$, i.e. $V_0=V_{i'}^{s'}=V_{s'}^{i'}$. Then

$$\alpha^{F(U_i\cap U_s)}_{G(V_0)}=\alpha^{F(U_{i'}\cap U_{s'})}_{G(V_{i'}^{s'}\cap V_{s'}^{i'})}=\gamma^{F(U_{i'}\cap U_{s'})}_{G(V_{i'}^{s'}\cap V_{s'}^{i'})}=\delta^{F(U_{i'}\cap U_{s'})}_{G(V_{i'}^{s'}\cap V_{s'}^{i'})}\in\delta. \tag{4.6}$$

QM3: If $(U_{i_1}\cap U_{s_1})\cap(U_{i_2}\cap U_{s_2})\neq\emptyset$, and γ satisfies QM3, we have the following commutative diagram.

$$\begin{array}{ccc} F(U_{i_1}\cap U_{s_1})\cap(U_{i_2}\cap U_{s_2})) & \overset{\gamma_{1,2}}{\to} & G(V_{i_1}^{s_1}\cap V_{s_1}^{i_1})\cap(V_{i_2}^{s_2}\cap V_{s_2}^{i_2})) \\ \rho\downarrow & & \downarrow\tau \\ F_1(U_{i_1}\cap U_{s_1}) & \overset{\gamma_1=\delta_1}{\to} & G_1(V_{i_1}^{s_1}\cap V_{s_1}^{i_1}). \end{array}$$

Since $V_{i_1}^{s_1}\cap V_{i_2}^{s_2}$ and $V_{s_1}^{i_1}\cap V_{s_2}^{i_2}$ are admissible, we have $\gamma_{1,2}=\delta^{F(U_{i_1}\cap U_{s_1})\cap(U_{i_2}\cap U_{s_2}))}_{G(V_{i_1}^{s_1}\cap V_{s_1}^{i_1})\cap(V_{i_2}^{s_2}\cap V_{s_2}^{i_2}))}$. So QM3 holds for δ. Now the δ is a quasi-morphism on A. □

The proposition follows from Lemma 4.4 and Lemma 4.5. □

Corollary 4.6 *The relation R (4.3) is an equivalence relation.*

Proof It follows from Proposition 4.3. □

Definition 4.7 *An equivalence class of a quasi-morphism is called a morphism. Set $\alpha=(\alpha^{F(V_i)}_{G(V_i)}):\langle F,\rho\rangle_T\to\langle G,\tau\rangle_W$ for the equivalence class of α (still denoted by α).*

4.2. A composition of two morphisms Given two morphisms α and β:

$$\alpha=(\alpha^{F(U_i)}_{G(V_i)}):\langle F,\rho\rangle_{B_\alpha}\to\langle G,\tau\rangle_{B_\beta},\quad \beta=(\beta^{G(V_s)}_{H(W_s)}):\langle G,\tau\rangle_{B_\beta}\to\langle H,\sigma\rangle,$$

for $\alpha^{F(U_i)}_{G(V_i)}$, we have $V_i=\cup_s(V_i\cap V_s)$. By AS2, $G(V_i)=\cup_s G(V_i\cap V_s)$ and $\rho: G(V_i\cap V_s)\hookrightarrow G(V_i)$ is the inclusion map. By QM3, there exists an open set $W_{si}\subseteq W_s$ such that $\beta^{G(V_i\cap V_s)}_{H(W_{si})}$ is well-defined. Since $(\alpha^{F(U_i)}_{G(V_i)})^{-1}(G(V_i\cap V_s))\subseteq F(U_i)$ is open, there exists an open set $U_i^s\subseteq U_i$ and $(\alpha^{F(U_i)}_{G(V_i)})^{-1}(G(V_i\cap V_s))=F(U_i^s)\hookrightarrow F(U_i)$ by AS1. Thus by QM3 we have an open set $V_i^s\subseteq V_i$ such that the following diagram is commutative.

$$F(U_i^s)\overset{\alpha}{\to}G(V_i^s)\hookrightarrow G(V)=F(U_i^s)\hookrightarrow F(U_i)\overset{\alpha}{\to}G(V_i).$$

Thus $\mathrm{Im}(\alpha^{F(U_i^s)}_{G(V_i^s)})\subseteq\alpha^{F(U_i)}_{G(V_i)}(F(U_i^s))\subseteq G(V_i\cap V_s)$, and by AS2

$$Im\alpha^{F(U_i^s)}_{G(V_i^s)}\subseteq G(V_i^s)\cap G(V_i\cap V_s)\subseteq G(V_i^s\cap(V_i\cap V_s)). \tag{4.7}$$

Hence $\alpha^{F(U_i^s)}_{G(V_i^s\cap(V_i\cap V_s))}$ is well-defined by QM2, so is $\alpha^{F(U_i^s)}_{G(V_i\cap V_s)}$. The composition $\beta^{G(V_i\cap V_s)}_{H(W_{si})}\alpha^{F(U_i^s)}_{G(V_i\cap}$ is well-defined. We obtain a set $\Sigma_1=\{\beta^{G(V_i\cap V_s)}_{H(W_{si})}\alpha^{F(U_i^s)}_{G(V_i\cap V_s)}|$ all i and all $s\}$. If $U\subseteq U_i^s$ is open, then by QM3 there exists $\alpha^{F(U)}_{G(V)}$ with $V \subseteq V_i \cap V_s \subseteq V_s$ so that $\beta^{G(V)}_{H(W)}$ is well-defined for $W \subseteq W_s$. Hence the composition $\beta^{G(V)}_{H(W)}\alpha^{F(U)}_{G(V)}$ is well-defined. Let

$$\Sigma = \{x = \beta^{G(V)}_{H(W)}\alpha^{F(U)}_{G(V)} | \ \forall U \subseteq U_i^s \text{ is nonempty open subset for all } i, s, \qquad (4.8)$$

$$\text{and } V, W \text{ such that } x \text{ is well-defined}\},$$

then $\Sigma_1 \subseteq \Sigma \neq \emptyset$.

Lemma 4.8 $\Sigma : \langle F, \rho\rangle \cdots \to \langle H, \sigma\rangle$ *is a quasi-morphism.*

Proof Let π be $\{U_i^s |$ for all $i, s\}$. In fact, we have that

$$F(U_i) = (\alpha^{F(U_i)}_{G(V_i)})^{-1}(G(V_i)) = \cup_s(\alpha^{F(U_i)}_{G(V_i)})^{-1}(G(V_i \cap V_s)) = \cup_s F(U_i^s), \qquad (4.9)$$

and $\cup_s U_i^s \subseteq U_i$. Since both $\rho^{F(\cup_s U_i^s)}_{F(U_i)}$ and $\rho^{F(U_i^s)}_{F(\cup_s U_i^s)}$ are inclusion maps by AS1,

$$F(U_i) = \cup_s F(U_i^s) \subseteq F(\cup_s U_i^s) \subseteq F(U_i), \qquad (4.10)$$

So we have $F(\cup_s U_i^s) = F(U_i)$ by (4.9) and (4.10), hence $\rho^{F(\cup_s U_i^s)}_{F(U_i)} = 1_{F(U_i)}$. We obtain $\cup_s U_i^s = U_i$ by AS2 (i) (b). Therefore π is a sub-basis. Let $T' = \{U | \emptyset \neq U \subseteq U_i^s$ is open, $U_i^s \in \pi\}$, then T' is a (c)-basis.

It is easy to show that QM1 and QM2 hold for Σ. Now we observe that $T' \subseteq B_\alpha$ and $V \in B_\beta$ if $\beta^{G(V)}_{H(W)}\alpha^{F(U)}_{G(V)} \in \Sigma$, so Σ satisfies QM3 because both α and β satisfy QM3. Therefore $\Sigma : \langle F, \rho\rangle \cdots \to \langle H, \sigma\rangle$ is a quasi-morphism on T'. □

Lemma 4.9 *The equivalence class of* Σ *(the morphism* Σ*) is independent of selections of representatives of* α *and* β.

Proof Suppose that $\alpha \overset{R}{\sim} \alpha'$ and $\beta \overset{R}{\sim} \beta'$. If $\gamma \subseteq \alpha \cap \alpha'$ and $\delta \subseteq \beta \cap \beta'$, then

$$\{\delta^{G(V)}_{H(W)}\gamma^{F(U)}_{G(V)}\} \subseteq \{\beta^{G(V)}_{H(W)}\alpha^{F(U)}_{G(V)}\} \cap \{\beta'^{G(V)}_{H(W)}\alpha^{F(U)}_{G(V)}\} \cap \{\beta^{G(V)}_{H(W)}\alpha'^{F(U)}_{G(V)}\} \cap \{\beta'^{G(V)}_{H(W)}\alpha'^{F(U)}_{G(V)}\},$$

so $\beta\alpha, \beta'\alpha, \beta\alpha'$ and $\beta'\alpha'$ have their common reductions. □

Definition 4.10 *The morphism* Σ *is called the composition of* α *and* β.

4.3. A category $\mathcal{AS}$

Proposition 4.11 *There is a category* $\mathcal{AS}$ *of all anti-sheaves:*
Objects: all anti-sheaves;
Arrows: all morphisms in the Definition 4.7;
Composition: the compositions in the Definition 4.10.

Proof It amounts to verify the associativity and unit law. Clearly $1_{\langle F,\rho\rangle} = (1_{F(U)})$ the Unit Law holds.

Associativity. Given a configuration $\langle F,\rho\rangle \xrightarrow{\alpha} \langle G,\tau\rangle \xrightarrow{\beta} \langle H,\sigma\rangle \xrightarrow{\gamma} \langle A,\delta\rangle$, we have

$$\beta\alpha = \{\beta_{H(W)}^{G(V)}\alpha_{G(V)}^{F(U)} | \forall \text{ open set } U \subseteq U_i^s, U_i^s \in \{U_i^s\}_{i,s}\},$$

$$\gamma\beta = \{\gamma_{A(O)}^{H(W)}\beta_{H(W)}^{G(V)} | \forall \text{ open set } V \subseteq V_s^t, V_s^t \in \{V_s^t\}_{s,t}\},$$

$$\gamma(\beta\alpha) = \{\gamma_{A(O)}^{H(W)}(\beta_{H(W)}^{G(V)}\alpha_{G(V)}^{F(U)}) | \forall \text{ open set } U \subseteq U_{i,s}^t\},$$

$$(\gamma\beta)\alpha = \{(\gamma_{A(O)}^{H(W)}\beta_{H(W)}^{G(V)})\alpha_{G(V)}^{F(U)} | \forall \text{ open set } U \subseteq U_i^{s,t}\},$$

where $U_i = \cup_s U_i^s = \cup_{s,t} U_i^{s,t} = \cup_{s,t} U_{i,s}^t$. Let $U_{i,s,t} = U_i^{s,t} \cap U_{i,s}^t$ if it is not empty. Then $U_i = \cup_{s,t} U_{i,s,t}$, so

$$B = \{U | \emptyset \neq U \subseteq U_i \subseteq U_{i,s,t} \in \{U_{i,s,t}\}_{i,s,t}\}, \tag{4.11}$$

is a (c)-basis. For each $U \in B$, we have $U \subseteq U_i^{s,t} \cap U_{i,s}^t$. If $V \subseteq V_i^{s,t}$ and $V_1 \subseteq V_{i,s}^t$, then $\alpha_{G(V)}^{F(U)}$ and $\alpha_{G_1(V_1)}^{F(U)}$ are well-defined by QM3 since $\alpha_{G(V_i^{s,t})}^{F(U_i^{s,t})}$ and $\alpha_{G_1(V_{i,s}^t)}^{F(U_{i,s}^t)}$ are well-defined. So $V \cap V_1 \neq \emptyset$ and there exist $\alpha_{G(V\cap V_1)}^{F(U)}$ and $\alpha_{G_1(V\cap V_1)}^{F(U)}$. Let $\lambda = \{\gamma_{A(O)}^{H(W)}\beta_{H(W)}^{G(V\cap V_1)}\alpha_{G(V\cap V_1)}^{F(U)}\}$, then

$$\lambda \subseteq \gamma(\beta\alpha) \cap (\gamma\beta)\alpha, \tag{4.12}$$

Thus $\gamma(\beta\alpha) \overset{R}{\sim} (\gamma\beta)\alpha$ by (4.12) (or $(\gamma(\beta\alpha),(\gamma\beta)\alpha) \in R$) because λ is a quasi-morphism. The result follows. □

From the above discussion, it is clear that we have the following.

Proposition 4.12 *If both B' and B are (c)-bases of T such that $B' \subseteq B$, then we have*

$$\langle F,\rho\rangle_{B'} \subset \langle F,\rho\rangle_B \quad \text{and} \quad \langle F,\rho\rangle_{B'} \cong \langle F,\rho\rangle_B \quad \text{in } \mathcal{AS}.$$

5. Adjoint equivalence

5.1. A binary relation Given a C^r n-manifold (M,Φ) with a (c)-basis B. To each $U \in B$ we assign a set $S_U = \{\phi | \forall(\phi,U) \in \Phi\}$ and a set $R_U = \{\phi(U) | \phi \in S_U\}$. Let $\alpha_U : S_U \to R_U; \phi \mapsto \phi(U)$. When $V \subseteq U$, we put $\rho_{\phi(U)}^{\phi'(V)} = \phi(\phi')^{-1}$, i.e. $\rho_{\phi(U)}^{\phi'(V)}$ is a symbol with its value $\phi(\phi')^{-1}$. Then we get a C^r n anti-sheaf $\langle\Phi,\rho\rangle_B$.

Definition 5.1 $\mathbf{U} : \mathcal{C}^r \to \mathcal{AS}$ *be a forgetful functor:* $\mathbf{U}(M,\Phi) = \langle\Phi,\rho\rangle_B$ *for* $(M,\Phi) \in ob\mathcal{C}^r$; $\mathbf{U}f = \{$*all* C^r *local representations of* $f\}$ *for* $f : (M,\Phi) \to (N,\Psi) \in Arr\mathcal{C}^r$.

Clearly, $\mathbf{U}f$ is a morphism from $\mathbf{U}(M,\Phi)$ to $\mathbf{U}(N,\Psi)$ in $\mathcal{AS}$. Sometimes we write $\mathbf{U}f = \{\psi_V f \phi_U^{-1}\}$ for short.

Given an anti-sheaf $\langle F,\rho\rangle_T$, set $\tilde{X} = \{\langle F(U),x\rangle | \forall F(U) \in \langle F,\rho\rangle_T, \forall x \in F(U)\}$. We define a binary relation $\mathcal{R}$ in the set $\tilde{X}$ as follows:

$$\langle\langle F(U),x\rangle,\langle F'(V),y\rangle\rangle \in \mathcal{R} \quad \text{if and only if } U\cap V \neq \emptyset \text{ and } \rho_{F'(V)}^{F(U\cap V)}x = y.$$

Lemma 5.2 $\langle\langle F(U),x\rangle,\langle F'(U'),y\rangle \in \mathcal{R}$ *if and only if* $y = \rho_{F'(U\cap U')}^{F(U\cap U')}x$ *and* $x = \rho_{F(U\cap U')}^{F'(U\cap U')}y$.

Proof ($\Rightarrow$:) $y = \rho^{F(U\cap U')}_{F'(U')} x = \rho^{F'(U\cap U')}_{F'(U')} \rho^{F(U\cap U')}_{F'(U\cap U')} x = \rho^{F(U\cap U')}_{F'(U\cap U')} x$. Since $y \in F'(U \cap U')$, we get $\rho^{F'(U\cap U')}_{F(U\cap U')} y = \rho^{F'(U\cap U')}_{F(U\cap U')} \rho^{F(U\cap U')}_{F'(U\cap U')} x = x$.

($\Leftarrow$:) $y = \rho^{F(U\cap U')}_{F'(U\cap U')} x = \rho^{F'(U\cap U')}_{F'(U')} \rho^{F(U\cap U')}_{F'(U\cap U')} x = \rho^{F(U\cap U')}_{F'(U')} x$.

Proposition 5.3 *The relation $\mathcal{R}$ is an equivalence relation.*

Proof (1) If $\langle\langle F(U), x\rangle, \langle F'(V), y\rangle\rangle \in \mathcal{R}$, by Lemma 5.2,

$$x = \rho^{F'(U\cap V)}_{F(U\cap V)} y = \rho^{F(U\cap V)}_{F(U)} \rho^{F'(U\cap V)}_{F(U\cap V)} y = \rho^{F'(U\cap V)}_{F(U)} y.$$

Thus $\langle\langle F'(V), y\rangle, \langle F(U), x\rangle\rangle \in \mathcal{R}$.

(2) If $\langle\langle F(U), x\rangle, \langle F'(V), y\rangle\rangle \in \mathcal{R}$, and $\langle\langle F'(V), y\rangle, \langle F''(W), z\rangle\rangle \in \mathcal{R}$, by Lemma 5.2 and the axioms AS1 and AS2, $y \in F'(U \cap V) \cap F'(V \cap W) = F'(U \cap V \cap W)$, thus

$$z = \rho^{F'(V\cap W)}_{F''(V\cap W)} y = \rho^{F'(U\cap V\cap W)}_{F''(V\cap W)} y = \rho^{F'(U\cap V\cap W)}_{F''(U\cap V\cap W)} y. \tag{5.1}$$

Since we have

$$\begin{aligned} x &= \rho^{F'(U\cap V)}_{F(U\cap V)} y = \rho^{F'(U\cap V)}_{F(U\cap V)} \rho^{F'(U\cap V\cap W)}_{F'(U\cap V)} y \\ &= \rho^{F'(U\cap V\cap W)}_{F(U\cap V)} y = \rho^{F(U\cap V\cap W)}_{F(U\cap V)} \rho^{F'(U\cap V\cap W)}_{F(U\cap V\cap W)} y \\ &= \rho^{F'(U\cap V\cap W)}_{F(U\cap V\cap W)} y, \end{aligned}$$

thus $x \in F(U \cap V \cap W)$. By Lemma 5.2,

$$\rho^{F(U\cap V\cap W)}_{F'(U\cap V\cap W)} x = y. \tag{5.2}$$

Therefore

$$\begin{aligned} z &= \rho^{F'(U\cap V\cap W)}_{F''(U\cap V\cap W)} y && (5.1) \\ &= \rho^{F'(U\cap V\cap W)}_{F''(U\cap V\cap W)} \rho^{F(U\cap V\cap W)}_{F'(U\cap V\cap W)} x && (5.2) \\ &= \rho^{F(U\cap V\cap W)}_{F''(U\cap W)} x && (\text{AP2(ii)}) \\ &= \rho^{F(U\cap W)}_{F''(U\cap W)} x, \end{aligned}$$

we obtained that $\langle\langle F(U), x\rangle, \langle F''(W), z\rangle\rangle \in \mathcal{R}$. $\square$

The equivalence class of $\langle F(U), x\rangle$ is denoted by $\overline{\langle F(U), x\rangle}$.

Lemma 5.4 $\overline{\langle F(U), x\rangle} = \overline{\langle F(U), x'\rangle}$ *if and only if* $x = x'$.

Lemma 5.5 $\{\overline{\langle F(U), x\rangle} | \forall x \in F(U)\} = \{\overline{\langle F'(U), y\rangle} | \forall y \in F'(U)\}$.

Proof Let $y = \rho^{F(U)}_{F'(U)} x$. Then $\langle F(U), x\rangle \overset{\mathcal{R}}{\sim} \langle F'(U), y\rangle$, and

$$\{\overline{\langle F(U), x\rangle} | \forall x \in F(U)\} \subseteq \{\overline{\langle F'(U), y\rangle} | \forall y \in F'(U)\}.$$

Let $x = \rho_{F(U)}^{F'(U)} y$, then the proof is complete. $\square$

Lemma 5.6 *$\{\overline{\langle F'(V), x\rangle} | \forall x \in F'(V)\} \subseteq \{\overline{\langle F(U), x\rangle} | \forall x \in F(U)\}$ if and only if $\emptyset \neq V \subseteq U$.*

Proof ($\Rightarrow$:) For each $x \in F'(V)$, $\langle F'(V), x\rangle \overset{\mathcal{R}}{\sim} \langle F(U), y_x\rangle$ and $y_x = \rho_{F(U)}^{F'(U\cap V)} x$, we have $x \in F'(V \cap U)$ and $F'(V) = F'(U \cap V)$. By AP2 (i), $\rho_{F'(V)}^{F'(U\cap V)} = 1$ and $U \cap V = V$, so $V \subseteq U$.

($\Leftarrow$:) Since $x = \rho_{F(U)}^{F(V)} x$ for $x \in F(V)$, we have $\langle F(V), x\rangle \overset{\mathcal{R}}{\sim} \langle F(U), x\rangle$. So

$$\{\overline{\langle F(V), x\rangle} | \forall x \in F(V)\} \subseteq \{\overline{\langle F(U). x\rangle} | \forall x \in F(U)\}.$$

Therefore the result follows from Lemma 5.5. $\square$

Lemma 5.7 *If $\emptyset \neq U_1 \subseteq U$, then $\overline{\langle F(U), x\rangle} = \overline{\langle F(U_1), x\rangle}$ for $x \in F(U_1)$.*

Proof For $x = \rho_{F(U)}^{F(U_1)} x$, the result follows. $\square$

Lemma 5.8 *If $U \cap U' \neq \emptyset$, then*

$\{\overline{\langle F(U), x\rangle} |\ \forall x \in F(U)\} \cap \{\overline{\langle F'(U'), y\rangle} |\ \forall y \in F'(U')\} = \{\overline{\langle F(U \cap U'), x\rangle} |\ \forall x \in F(U\cap U')\}$.

Proof By Lemma 5.6 and Lemma 5.7, we have

$$\begin{aligned}\{\overline{\langle F(U), x\rangle} |\ \forall x \in F(U \cap U')\} &= \overline{\langle F(U \cap U'), x\rangle} |\ \forall x \in F(U \cap U')\} \\ &\subseteq \{\overline{\langle F(U), x\rangle} |\ \forall x \in F(U)\} \cap \{\overline{\langle F'(U'), y\rangle} |\ \forall y \in F'(U')\}.\end{aligned}$$

Let $\overline{\langle F(U), x_0\rangle} = \overline{\langle F'(U'), y_0\rangle} \in \{\overline{\langle F(U), x\rangle} |\ \forall x \in F(U)\} \cap \{\overline{\langle F'(U'), x\rangle} |\ \forall x \in F'(U')\}$. Thus $x_0 \in F(U \cap U')$ and $\overline{\langle F(U), x_0\rangle} \in \{\overline{\langle F(U), x\rangle} |\ \forall x \in F(U \cap U')\}$. $\square$

5.2. A C^r anti-sheaf manifold

Definition 5.9 $X = \{\overline{\langle F(U), x\rangle} |\ \forall x \in F(U), \forall U \in T, \forall F(U) \in \langle F, \rho\rangle_T\}$,

$$B = \{\{\overline{\langle F(U), x\rangle} |\ \forall x \in F(U)\} |\ \forall U \in T, \forall F(U) \in \langle F, \rho\rangle_T\}.$$

Then B is a basis of X by Lemma 5.8. Let τ be the topology generated by B. We have a topological space (X, τ).

Given a map $\phi_{F(U)} : \{\overline{\langle F(U), x\rangle} | \forall x \in F(U)\} \to F(U)$ ($\overline{\langle F(U), x\rangle} \mapsto x$). By Lemma 5.4, $\phi_{F(U)}$ makes sense. By AS1 and Lemma 5.7, $\phi_{F(U)}$ is a bijective continuous open map. Therefore $(\phi_{F(U)}, \{\overline{\langle F(U), x\rangle} | \forall x \in F(U)\})$ is a chart.

If $G = \{\overline{\langle F(U), x\rangle} |\ \forall x \in F(U)\} \cap \{\overline{\langle F_1(U_1), y\rangle} |\ \forall y \in F_1(U_1)\} \neq \emptyset$, then by Lemma 5.8, $G = \{\overline{\langle F(U \cap U_1), x\rangle} |\ \forall x \in F(U \cap U_1)\} = \{\overline{\langle F_1(U \cap U_1), y\rangle} |\ \forall y \in F_1(U \cap U_1)\}$. We get

$$\begin{aligned}\phi_{F_1(U_1)}\phi_{F(U)}^{-1}|_{F(U\cap U_1)}(x) &= \phi_{F_1(U_1)}(\overline{\langle F(U \cap U_1), x\rangle}) \\ &= \phi_{F_1(U_1)}(\overline{\langle F(U \cap U_1), \rho_{F_1(U\cap U_1)}^{F(U\cap U_1)} x\rangle}) \\ &= \phi_{F_1(U_1)}(\overline{\langle F_1(U_1), \rho_{F_1(U_1)}^{F(U\cap U_1)} x\rangle}) \\ &= \rho_{F_1(U_1)}^{F(U\cap U_1)} x,\end{aligned}$$

therefore the atlas $\{(\phi_{F(U)}, \{\overline{\langle F(U), x\rangle} |\ \forall x \in F(U)\})|\ \forall F(U) \in \langle F, \rho\rangle_T\}$ defines a C^r n-manifold (X, Φ) which is called a C^r *anti-sheaf manifold* and is denoted by $AM(\langle F, \rho\rangle_T)$. Also $\overline{B} = \{\{\overline{\langle F(U), x\rangle} |\ \forall x \in F(U)\}|\ \forall F(U) \in \langle F, \rho\rangle_T\}$ is a (c)-basis of $\mathbf{U}AM(\langle F, \rho\rangle_T)$.

Definition 5.10 *In $\mathcal{AS}$, a morphism $\eta : \langle F, \rho\rangle_T \to \mathbf{U}AM(\langle F, \rho\rangle_T)$ is defined as follows: Given an open set $U \in T$, let $\{\overline{\langle F(U), x\rangle} |\ \forall x \in F(U)\}$ be a corresponding open set; $\eta^{F(U)}_{F_1(V)} : F(U) \to F_1(V)$ be the map $\rho^{F(U)}_{F_1(V)}$ if $\{\overline{\langle F(U), x\rangle} |\ \forall x \in F(U)\} \subseteq \{\overline{\langle F_1(V), x\rangle} |\ \forall x \in F_1(V)\}$.*

For $U \subseteq V$ the definition of η is well-defined by Lemma 5.6. We also have

$$\eta^{F(U)}_{F_1(V)} x = \rho^{F(U)}_{F_1(V)} x = \rho^{F_1(U)}_{F_1(V)} \rho^{F(U)}_{F_1(U)} x = \eta^{F(U)}_{F_1(U)} x.$$

Let $\eta = \{\text{all } \eta^{F(U)}_{F_1(U)}\}$. It is easy to check that η is a morphism on T.

Definition 5.11 *A morphism Θ on $\overline{B}$ is defined as the following:*

$$\Theta = \{\text{all } \theta^{F'(U)}_{F(V)}\} : \mathbf{U}AM(\langle F, \rho\rangle_T) \to \langle F, \rho\rangle_T,$$

for $U \subseteq V \in T$, $\theta^{F'(U)}_{F(V)} = \rho^{F'(U)}_{F(V)}$.

It is easy to prove that Θ is a morphism on $\overline{B}$ and that $\Theta\eta = 1_{\langle F, \rho\rangle_T}$ and $\eta\Theta = 1_{\mathbf{U}AM(\langle F, \rho\rangle_T)}$. Therefore we obtain

$$\mathbf{U}AM(\langle F, \rho\rangle_T) \cong \langle F, \rho\rangle_T \qquad \text{in } \mathcal{AS}. \tag{5.3}$$

Definition 5.12 *Given a C^r n-manifold $(Y, \Psi) \in ObC^r$ and a morphism $\alpha : \langle F, \rho\rangle_T \to \mathbf{U}(Y, \Psi)$ on B_1, we defined a map $h : AM(\langle F, \rho\rangle_T) \to (Y, \Psi)$,*

$$\overline{\langle F(U), x\rangle} \mapsto \psi_V^{-1} \alpha^{F(U)}_{\psi_V(V)} x \quad \text{for } U \in B_1. \tag{5.4}$$

Lemma 5.13 *The map h is well-defined and a C^r map.*

Proof For each $a \in X$ there exists an open set $W \in T$ such that $a = \overline{\langle F(W), x_0\rangle}$. Since B_1 is a (c) sub-basis of T, we have $F(W) = \cup_{i \in I} F(U_i)$ for $U_i \in B_1$. Hence $x_0 \in F(U_{i_0})$ for some $i_0 \in I$. We obtain that $a = \overline{\langle F(U_{i_0}), x_0\rangle}$ by Lemma 5.7.

Given $U_1, U \in B_1$ with $\overline{\langle F_1(U_1), y\rangle} = \overline{\langle F(U), x\rangle}$, then $x \in F(U \cap U_1)$. Set $\mathbf{U}(Y, \Psi) = \langle \Psi, r\rangle$. By QM3 we have the following commutative diagram.

$$\begin{array}{ccc} F(U \cap U_1) & \xrightarrow{\alpha^{F(U\cap U_1)}_{\psi(V\cap V_1)}} & \psi(V \cap V_1) \\ \rho^{F(U\cap U_1)}_{F_1(U_1)} \downarrow & & \downarrow r_1 = \psi'\psi^{-1} \\ F_1(U_1) & \xrightarrow{\alpha^{F_1(U_1)}_{\psi'(V_1)}} & \psi'(V_1). \end{array} \tag{5.5}$$

Hence $r_1 \alpha^{F(U\cap U_1)}_{\psi(V\cap V_1)} x = \alpha^{F_1(U_1)}_{\psi'(V_1)} y$ by (5.5). So

$$(\psi')^{-1} \alpha^{F_1(U_1)}_{\psi'(V_1)} y = \psi^{-1} \alpha^{F(U\cap U_1)}_{\psi(V\cap V_1)} x = \psi^{-1} \alpha^{F(U)}_{\psi_V(V)} x. \tag{5.6}$$

The well-definedness of h follows from (5.6). Note that $\phi_{F(U)}(\overline{\langle F(U), x\rangle}) = x$ and

$$\psi_V h \phi_{F(U)^{-1}} x = \psi_V(\psi^{-1}\alpha^{F(U)}_{\psi_V(V)} x) = \alpha^{F(U)}_{\psi_V(V)} x.$$

so h is, in fact, a C^r map. □

5.3. Adjunction

Note that $\mathbf{U}h = \{\alpha^{F(U)}_{\psi_V(V)} | \forall \alpha^{F(U)}_{\psi_V(V)}$ a C^r local representation of $h\}$ and the morphism $\alpha^{F(U)}_{\psi_V(V)} \eta^{F(U)}_{F(U)} = \alpha^{F(U)}_{\psi_V(V)}$. We have the following identity:

$$\mathbf{D}h \cdot \eta = \alpha,$$

where $\mathbf{D}$ is the diagonal functor. If $h_1 : AM(\langle F, \rho\rangle_T) \to (Y, \Psi)$ with $\mathbf{U}h_1 \cdot \eta = \alpha$, then

$$(\psi_{V'} h_1 \phi^{-1}_{F'(U')})(\eta^{F'(U')}_{F'(U')} x) = \alpha^{F'(U')}_{\psi_{V'}(V')} x,$$

so we obtain

$$h_1(\overline{\langle F'(U'), x\rangle}) = \psi^{-1}_{V'} \alpha^{F'(U')}_{\psi_{V'}(V')} x = h(\overline{\langle F'(U'), x\rangle}).$$

Therefore $h_1 = h$. The morphism η is a universal arrow from $\langle F, \rho\rangle_T$ to $\mathbf{U}$.

Proposition 5.14 *There exists an adjunction*

$$\langle AM, \mathbf{U}, \eta, \varepsilon\rangle : \mathcal{AS} \rightharpoonup \mathcal{C}^r.$$

Proof The result follows from the above and Theorem 2 (ii) in [7] page 81. □

Theorem 5.15 *For an adjunction $\langle F, G, \eta, \varepsilon\rangle : X \rightharpoonup A$:*

(i) F is faithful if and only if every component η_x of the unit η is monic,

(ii) F is full if and only if every η_x is a split epi.

Hence F is full and faithful if and only if each η_x is an isomorphism $x \cong GFx$ (see Theorem 1 in [7] p88).

By methods in [7] §IV 3, we can complete the proof by showing the following lemmas.

Lemma 5.16 *Let $f_* = (f_{*c})_{c \in ObA} : A(-, a) \dot{\to} A(-, b)$ be the natural transformation induced by an arrow $f : a \to b$ of A. Then for each $c \in ObA$, f_{*c} is monic if and only if f is monic; f_{*c} is epi if and only if f is a split epi, i.e. if and only if f has a right inverse.*

Lemma 5.17 *If $\langle T, G, \eta, \varepsilon\rangle : \mathcal{C} \rightharpoonup \mathcal{A}$ is an adjunction and if $\langle T', G, \eta', \varepsilon'\rangle : \mathcal{C} \rightharpoonup \mathcal{A}$ is an adjoint equivalence, then $\langle T, G, \eta, \varepsilon\rangle : \mathcal{C} \rightharpoonup \mathcal{A}$ is also an adjoint equivalence.*

Proof By [7] p 91, we know that $\langle T', G, \eta', \varepsilon'\rangle : \mathcal{C} \rightharpoonup \mathcal{A}$ is an adjoint equivalence, so is the adjunction $\langle G, T', (\varepsilon')^{-1}, (\eta')^{-1}\rangle : \mathcal{A} \rightharpoonup \mathcal{C}$. Hence both T' and G are full and faithful by Theorem 1 (iii) in [7] p 91. Since the functors T and T' are naturally isomorphic by Corollary 1 [7] p 83, the functor T is also full and faithful. Therefore both η and ε are natural isomorphisms by Theorem 1 [7] p88 and Theorem 5.15. The proof is complete. □

Theorem A *The adjunction* $\langle AM, \mathbf{U}, \eta, \varepsilon\rangle : \mathcal{AS} \rightharpoonup C^r$ *is an adjoint equivalence.*

Proof (1) $\mathbf{U}$ is faithful. Given two C^r maps $f, g : (X, \Phi) \to (Y, \Psi)$ with $\mathbf{U}f = \mathbf{U}g$. Let $h : \mathbf{U}(X, \Phi) \to \mathbf{U}(Y, \Psi)$ be their common reduction. Now we have

$$\mathbf{U}g = \{\psi_V g \phi_U^{-1}\} \quad \text{and} \quad \mathbf{U}f = \{\psi'_{V'} f (\phi'_{U'})^{-1}\}. \tag{5.7}$$

Let $B_h = \{U_i | i \in I\}$ be a basis of h. So

$$h = \{\psi_{V_i} f \phi_{U_i}^{-1}\} = \{\psi_{V_i} g \phi_{U_i}^{-1}\}, \tag{5.8}$$

and $\psi_{V_i} f \phi_{U_i}^{-1} = \psi_{V_i} g \phi_{U_i}^{-1}$. If $x \in U_i$, then $(\psi_{V_i} f \phi_{U_i}^{-1})(\phi_{U_i}(x)) = \psi_{V_i} f(x) = \psi_{V_i} g(x)$. Hence $f(x) = g(x)$, so $f = g$.

(2) $\mathbf{U}$ is full. Given a morphism $\alpha = \{\alpha^{\phi_U(U)}_{\psi_V(V)} | U \in B_\alpha\} : \mathbf{U}(X, \Phi) \to \mathbf{U}(Y, \Psi)$. For each $x \in X$, there exists an open set $U \in B_\alpha$ such that $x \in U$. Let $r : X \to Y$ be a map such that $x \mapsto \psi_V^{-1}(\alpha^{\phi_U(U)}_{\psi_V(V)}(\phi_U(U)(x)))$ for $x \in U$. The map r is well-defined. If $x \in U_1 \cap U, U, U_1 \in B_\alpha$, using the definition of $\mathbf{U}$ and Axiom QM3, we get the following commutative diagram:

$$\begin{array}{llllll}
\phi_U(x) \in \phi_U(U \cap U_1) & \xrightarrow{\rho_1} & \phi_1(U \cap U_1) & \xrightarrow{\alpha^{\phi_1(U\cap U_1)}_{\psi_1(V\cap V_1)}} & \psi_1(V \cap V_1) & \\
\downarrow & \rho_2 \swarrow & \downarrow & & \downarrow & \searrow \\
\phi_U(U) & & \phi_1(U_1) & \xrightarrow{\alpha^{\phi_1(U_1)}_{\psi_1(V_1)}} & \psi_1(V_1) & \\
\| & & & & & \searrow \sigma_1 \\
\phi_U(U) & \cdots & & \overset{\alpha^{\phi_U(U)}_{\psi_V(V)}}{\cdots} & & \to \psi_V(V).
\end{array} \tag{5.9}$$

where $\sigma_1 = \psi_V \psi_1^{-1}|_{\psi_1(V \cap V_1)}$, $\rho_1 = \phi_1 \phi_U^{-1}|_{\phi_U(U\cap U_1)}$ and $\rho_2 = \phi_U \phi_1^{-1}|_{\phi_1(U \cap U_1)}$. For $x \in U \cap U_1$,

$$\begin{aligned}
\alpha^{\phi_U(U)}_{\psi_V(V)}(\rho_2(\rho_1(\phi_U(x)))) &= \alpha^{\phi_U(U)}_{\psi_V(V)} \phi_U(x) \\
&= \sigma_1 \alpha^{\phi_1(U\cap U_1)}_{\psi_1(V \cap V_1)}(\rho_1(\phi_U(x))) \qquad (5.9) \\
&= (\psi_V \psi_1^{-1})|_{\psi_1(V\cap V_1)} \alpha^{\phi_1(U_1)}_{\psi_1(V_1)}(\phi_1(x)).
\end{aligned}$$

Hence we obtain that

$$\psi_V^{-1}(\alpha^{\phi_U(U)}_{\psi_V(V)} \phi_U(x)) = \psi_1^{-1}(\alpha^{\phi_1(U_1)}_{\psi_1(V_1)}(\phi_1(x))). \tag{5.10}$$

So r is well-defined by (5.10). We also have $(\psi_V r \phi_U^{-1})(\phi_U(x)) = \alpha^{\phi_U(U)}_{\psi_V(V)}(\phi_U(x))$, so $\psi_V r \phi_U^{-1}$ is C^r map. Clearly $\mathbf{U}r = \alpha$, so $\mathbf{U}$ is full.

(3) We have already proved that for any $\langle F, \rho\rangle_T \in \mathcal{AS}$ there exists a manifold $AM(\langle F, \rho\rangle_T)$ ObC^r such that $\eta : \langle F, \rho\rangle_T \cong \mathbf{U}AM(\langle F, \rho\rangle_T)$ in $\mathcal{AS}$. By Theorem 1 [7] p91 there exists a functor $L : \mathcal{AS} \to C^r$ such that $\langle L, \mathbf{U}, \eta', \varepsilon'\rangle : \mathcal{AS} \rightharpoonup C^r$ is an adjoint equivalence. The result follows from Proposition 5.14 and Lemma 5.17. □

Corollary 5.18 *Every C^r-manifold is C^r-diffeomorphic to its anti-sheaf manifold.*

Corollary 5.19 *If (X, Φ) and (Y, Ψ) are two C^r-manifolds, then*

$$(X, \Phi) \approx (Y, \Psi) \quad \textit{if and only if} \quad \mathbf{U}(X, \Phi) \cong \mathbf{U}(Y, \Psi).$$

Proof If $\mathbf{U}(X, \Phi) \cong \mathbf{U}(Y, \Psi)$ then $AM\mathbf{U}(X, \Phi) \approx AM\mathbf{U}(Y, \Psi)$. Corollary 5.18 shows that $(X, \Phi) \approx (Y, \Psi)$. □

Remarks 1. Theorem A shows that the set $\{\phi_j \phi_i^{-1}\}$ is the principal part of a C^r differential structure.

2. There are manifolds without smooth structures on $AM(\langle F, \rho \rangle_T)$ (see [1,2,9]), i.e., there are (c)-bases without a C^r anti-sheaf structure ($r \geq 1$) by Corollary 5.18.

6. A characterization of C^r-diffeomorphisms

Definition 6.1 *A semi-group G is called a weak group if the followings hold:*

1. *$G = \coprod_{i \in I} G_i$ is a disjoint union of all G_i, where G_i is a semi-group with its left unit l_i.*
2. *$G = \coprod_{j \in J} H_j$ is a disjoint union of all H_j, where H_j is a semi-group with its right unit r_j.*
3. *For each $g_i \in G_i$, there exists $h_j \in H_j$ such that $h_j g_i = r_j$ and $g_i h_j = l_i$ for some $j \in J$.*

Given a C^r n-manifold (X, Φ) with $\Phi = \{\phi_U(U) | U \in B\}$. Let $\langle \Phi, \rho \rangle_B = \mathbf{U}(X, \Phi)$. To each open set $U \in B$ we assign a set $G_r(U) = \{\rho^{\tilde{\phi}(U)}_{\phi''(U)}, \rho^{p_\phi(U)}_{p'_{\phi'}(U)}, \rho^{\phi(U)}_{p'_{\phi'}(U)}, \rho^{p_\phi(U)}_{\phi'(U)} | \forall \rho^{\tilde{\phi}(U)}_{\phi''(U)} \in \langle \Phi, \rho \rangle_B; \forall p, \phi, p'\phi'$ locally homeomorphism$\}$. A map $p : U \to \mathbf{R}^n$ is called a ϕ locally homeomorphism if it is a locally homeomorphism and for some $\phi \in \Phi, p(U) = \phi(U)$. The symbol $\rho^{p_\phi(U)}_{p'_{\phi'}(U)}$ is new with its value $\rho^{\phi(U)}_{\phi'(U)}$ and the symbol $\rho^{\phi(U)}_{p'_{\phi'}(U)}$ with its value $\rho^{\phi(U)}_{\phi'(U)}$.

The set $G_r(U)$ can be formed into a semi-group with an associative binary composition $\circ$ such that

$$\circ : G_r(U) \times G_r(U) \to G_r(U)$$

$$\langle \rho^{\phi'(U)}_{\phi(U)}, \rho^{\phi''(U)}_{\tilde{\phi}(U)} \rangle \mapsto \rho^{\phi''(U)}_{\phi(U)}, \quad \langle \rho^{\phi(U)}_{\phi'(U)}, \rho^{\tilde{p}_{\tilde{\phi}}(U)}_{\phi''(U)} \rangle \mapsto \rho^{\tilde{p}_{\tilde{\phi}}(U)}_{\phi'(U)}.$$

Then we have a weak group $G_r(U) = \coprod_{\forall \phi} G^{\phi}_U \coprod_{\forall p_\phi} G^{p_\phi}_U = \coprod_{\forall \phi} H^U_\phi \coprod \coprod_{\forall p_\phi} H^U_{p_\phi}$, where $G^{\phi}_U = \{\rho^{\phi'(U)}_{\phi(U)} | \forall \phi'\}$ is a semi-group with its left unit $\rho^{\phi(U)}_{\phi(U)}$ and $G^{p_\phi}_U = \{\rho^{\phi'(U)}_{p_\phi(U)} | \forall \phi'\}$ with its left unit $\rho^{p_\phi(U)}_{p_\phi(U)}$; $H^U_\phi = \{\rho^{\phi(U)}_{\phi'(U)} | \forall \phi'\}$ with its right unit $\rho^{\phi(U)}_{\phi(U)}$.

For $U' \subseteq U \in B$ ($U' \neq U$), there exists a homomorphism

$$f^U_{U'} : G_r(U) \to G_r(U') : \quad \rho^{\phi(U)}_{\phi_1(U)} \mapsto \rho^{\phi(U')}_{\phi_1(U')}, \rho^{\phi(U)}_{p_{\phi_1}(U)} \mapsto \rho^{\phi(U')}_{p_{\phi_1}(U')}. \tag{6.1}$$

Since $f^{U_1}_{U'} \cdot f^U_{U_1} = f^U_{U'}$ and let $f^U_U = id_{G_r(U)}$, we get a presheaf $[G_\phi, f]_B$ of the weak group on B. We show that the presheaf $[G_\phi, f]_B$ satisfies the condition (M) and weak (G) (see

[8] p 14 and Definition 6.6).

Lemma 6.2 *The presheaf $[G_\phi, f]_B$ satisfies the condition (M).*

Proof Suppose that $U = \cup_{i\in\Lambda} U_i$ for $U, U_i \in B$, and $s, s' \in G_r(U)$ such that $f^U_{U_i} s = f^U_{U_i} s', \forall i \in \Lambda$. Let $s = \rho^{\phi(U)}_{p'_{\phi'}(U)}$ and $s' = \rho^{\tilde{p}_{\tilde{\phi}}(U)}_{\tilde{\phi}(U)}$ by the definition of $G_r(U)$. Thus $\rho^{\phi(U_i)}_{p'_{\phi'}(U_i)}$ and $\rho^{\tilde{p}_{\tilde{\phi}}(U_i)}_{\tilde{\phi}(U_i)}$ are the same symbol, so $s = s'$. □

Let $U = \cup_{\lambda\in\Lambda} U_\lambda$ with U and $U_\lambda \in B$. Given a family $(s_\lambda)_{\lambda\in\Lambda}$ with $\forall\lambda \in \Lambda, s_\lambda \in G_r(U_\lambda)$ and $\forall\lambda,\mu \in \Lambda, U_\lambda \cap U_\mu \neq \emptyset$ such that $f^{U_\lambda}_{U_\lambda\cap U_\mu}(s_\lambda) = f^{U_\lambda}_{U_\lambda\cap U_\mu}(s_\mu), \forall\lambda,\mu \in \Lambda$. Let $s_\lambda = \rho^{\phi_\lambda(U_\lambda)}_{\phi'_\lambda(U_\lambda)}$ and $s_\mu = \rho^{\phi_\mu(U_\mu)}_{\phi'_\mu(U_\mu)}$, then $\phi_\lambda|_{U_\lambda\cap U_\mu} = \phi_\mu|_{U_\lambda\cap U_\mu}$.

For the set $\{\phi_{\lambda(U_\lambda)} | \lambda \in \Lambda\}$, we define a map $h : U \to \mathbf{R}^n$ such that for all $x \in U$, $x \mapsto \phi_\lambda(x)$ if x also lies in U_λ. It is well-defined due to $\phi_\lambda|_{U_\lambda\cap U_\mu} = \phi_\mu|_{U_\lambda\cap U_\mu}$. Since $h|_{U_\lambda} = \phi_\lambda : U_\lambda \to \phi_\lambda(U_\lambda)$, we have that $h(U) = \cup_{\lambda\in\Lambda}\phi_\lambda(U_\lambda)$ is an open set of $\mathbf{H}$, so $h(U) \in Ob\theta^r_n$. Note that h is a **local diffeomorphism**, not necessary a local coordinate.

Let $T \in (\theta^r_n)^J$ (like the functor F in Proposition 2.2) and $\phi_{\lambda\mu} = \phi_\lambda|_{U_\lambda\cap U_\mu}$. We have $\eta_\lambda : \phi_\lambda(U_\lambda) \to h(U)$ is defined by $\eta_\lambda(x) = h(\phi_\lambda^{-1}(x))$. Then $\eta_\lambda(x) = x$. Let $\eta_{\lambda\mu} : \phi_{\lambda\mu}(U_\lambda \cap U_\mu) \to h(U)$ be the map $x \mapsto \phi_\lambda|_{U_\lambda\cap U_\mu}(\phi^{-1}_{\lambda\mu}(x))$.

Proposition 6.3 $\eta = (\eta_\lambda)_{\lambda\in\Lambda} : T \dot{\to} \mathbf{D}h(U)$*, is a natural transformation.*

Proof Let $i(t) = (\phi_\lambda|_{U_\lambda\cap U_\mu}, \phi^{-1}_{\lambda\mu})(t)$ for $t \in \phi_{\lambda\mu}(U_\lambda \cap U_\mu)$. We have

$$\begin{aligned}\eta_\mu(\phi_\mu\phi^{-1}_{\lambda\mu}(t)) &= \eta_\mu(\phi_\mu\phi_\lambda^{-1}|_{U_\lambda\cap U_\mu}(i(t))) = \phi_\mu\phi_\lambda^{-1}|_{U_\lambda\cap U_\mu}(i(t)) \\ &= i(t) = \eta_{\lambda\mu}(t).\end{aligned}$$

Similarly we obtain that $\eta_\lambda(\phi_\lambda\phi^{-1}_{\lambda\mu}(i(t))) = \eta_{\lambda\mu}(t)$. So η is a natural transformation. □

Definition 6.4 *Given an open set $V \in Ob\theta^r_n$ and a natural transformation $g = (g_\lambda)_{\lambda\in\Lambda} : T \dot{\to} \mathbf{D}V$ with $g_\lambda \in C^r$, we define a map $m : h(U) \to V$ such that $x \mapsto g_\lambda(x)$ for $x \in \phi_\lambda(U_\lambda)$.*

Lemma 6.5 *The map m is well-defined and a C^r map.*

Proof Suppose that $x \in \phi_\mu(U_\mu)$ and $U_\lambda \cap U_\mu \neq \emptyset$. Note that g is a natural transformation. The following diagram is commutative:

$$\begin{array}{lll}\phi_{\lambda\mu}(U_\lambda \cap U_\mu) & & \\ \downarrow \phi_\lambda\phi^{-1}_{\lambda\mu} & & \searrow g_{\lambda\mu} \\ \phi_\lambda(U_\lambda) & \xrightarrow{g_\lambda} & V.\end{array}$$

Hence $g_{\lambda\mu}(x) = g_\lambda(x)$, similarly $g_{\lambda\mu}(x) = g_\mu(x)$. Therefore the map m makes sense and it is clear that m is a C^r map. □

Since $m(\eta_\lambda(x)) = g_\lambda(\eta_\lambda(x)) = g_\lambda(x)$, we get

$$\mathbf{D}m \cdot \eta = g. \tag{6.2}$$

If $m' : h(U) \to V \in C^r$ satisfies $\mathbf{D}m' \cdot \eta = g$, then $m'(x) = m'(\eta_\lambda(x)) = g_\lambda(x), \forall x \in \phi_\lambda(U_\lambda)$, so $m = m'$. Therefore we obtain the limit property

$$\lim_{\to} T = h(U). \tag{6.3}$$

By Proposition 2.2 there exists a chart $(U, \phi) \in \Phi$ such that $h(U) = \phi(U)$. A similar discussion for the set $\{\phi'_\lambda(U_\lambda) | \lambda \in \Lambda\}$ shows that there exists a map $h' : U \to \mathbf{R}^n$ such that $h'(U) = \phi'(U)$ for some chart $(U, \phi') \in \Phi$ and $h'|_{U_\lambda} = \phi'_\lambda$.

Definition 6.6 *Let* $s = \rho^{h_\phi(U)}_{h'_{\phi'}(U)}$. *Then* $s \in G_r(U)$,

$$(\rho^{h(U)}_{h'(U)} \mapsto \rho^{h(U_\lambda)}_{h'(U_\lambda)}) = (\rho^{\phi(U)}_{\phi'(U)} \mapsto \rho^{\phi(U_\lambda)}_{\phi'(U_\lambda)}) = f^U_{U_\lambda},$$

by (6.1) and $f^U_{U_\lambda}(s) = s_\lambda$. *Such a presheaf is called a W.(G)-sheaf. I.e. A presheaf satisfies condition* (M) *and weak* (G).

Remark The $f^U_{U_\lambda}$ in Definition 6.6 is defined similarly as (6.1) by using $h(U)$ instead of $\phi(U)$. Since $h(U) = \phi(U)$, so the s satisfies $f^U_{U_\lambda}(s) = s_\lambda$. But $h \neq \phi$ in general. So it is a weak condition (G). The sheafification of the presheaf $[G_\phi, f]_B$ (c.f. [4]) is not the object for our purpose.

Proposition 6.7 *The presheaf* $[G_\phi, f]_B$ *is a W.(G)-sheaf.*

Definition 6.8 $[G_\phi, f]_B$ *is called an inherent W.(G)-sheaf of the manifold* (X, Φ) *or of the anti-sheaf* $\langle \Phi, \rho \rangle_B$. *Given two W.(G)-sheaves* $[G_F, f]_B$ *and* $[G_H, g]_D$, *we have the following definition. If there is a bijective map* $m : B \to D$ *preserving intersections such that for each* $U \in B$, $G_F(U) = G_H(m(U))$, *i.e., they possess the same elements and the same binary composition, then* $[G_F, f]_B$ *is called to be equal to* $[G_H, g]_D$ *under* m. *We denote by* $[G_F, f]_B \overset{m}{=} [G_H, g]_D$.

Given two manifolds (X, Φ) and (Y, Ψ) such that $(X, \Phi) \approx (Y, \Psi)$. Let $\mathbf{U}(X, \Phi) = \langle F, \rho \rangle_{B_F}$ and $\mathbf{U}(Y, \Psi) = \langle H, \tau \rangle_{B_H}$. Thus $\langle F, \rho \rangle_{B_F} \cong \langle H, \tau \rangle_{B_H}$ in $\mathcal{AS}$. We shall show that there exist two bases $B''_F \subseteq B_F$ and $Q \subseteq B_H$, and a bijective order preserving map $m : B''_F \to Q$ such that $[G_F, f]_{B''_F} \overset{m}{=} [G_H, h]_Q$, where $[G_F, f]_{B''_F}$ is the inherent W.(G)-sheaf of the anti-sheaf $\langle F, \rho \rangle_{B''_F}$ and $[G_H, h]_Q$ is the one of the anti-sheaf $\langle H, \tau \rangle_Q$.

Theorem 6.9 *The inherent W.(G)-sheaf is a differential invariant.*

Proof Given two morphisms $\langle F, \rho \rangle_{B_F} \xrightarrow{\alpha} \langle H, \tau \rangle_{B_H} \xrightarrow{\beta} \langle F, \rho \rangle_{B_F}$, with $\beta\alpha = 1_{\langle F, \rho \rangle_{B_F}}$ and $\alpha\beta = 1_{\langle H, \tau \rangle_{B_H}}$. Since $1_{\langle F, \rho \rangle_{B_F}} = \{1_{F(U)} | \forall U \in B_F, \forall F(U) \in \langle F, \rho \rangle_{B_F}\}$ and $\beta\alpha = 1_{\langle F, \rho \rangle_{B_F}}$, the quasi-morphisms $\{\beta^{H(V)}_{F(W)} \alpha^{F(U)}_{H(V)} | \forall U \in B_F\}$ and $\{1_{F(U)} | \forall U \in B_F\}$ have a common reduction. So there exists a (c)-basis $B'_F \subseteq B_F$ with $L = \{\beta^{H(V)}_{F(U)} \alpha^{F(U)}_{H(V)} = 1_{F(U)} | \forall U \in B'_F\}$. Hence we get a set

$$A = \{V | \forall U \in B'_F, \forall \beta^{H(V)}_{F(W)} \alpha^{F(U)}_{H(V)} \in L\}. \tag{6.4}$$

Similarly we have a (c)-basis $B'_H \subseteq B_H$, a set

$$S = \{\alpha^{F_1(U'')}_{H_1(V')} \beta^{H_1(V')}_{F_1(U'')} = 1_{H_1(V')} | \forall V' \in B'_H\}, \tag{6.5}$$

and a set $D = \{U' | \forall V' \in B'_H, \forall \alpha^{F_1(U'')}_{H_1(V')} \beta^{H_1(V')}_{F_1(U'')} \in S\}$. For each $V \in A$ and $V = \cup_{V' \in B'_H}(V \cap V')$, we have $H(V) = \cup_{V' \in B'_H} H(V \cap V')$ by AS2. By QM1 and AS1, we have $(\alpha^{F(U)}_{H(V)})^{-1}(H(V \cap V')) = F(U')$ for a unique open set $U' \subseteq U$. Then we get a set $E = \{U' | \forall U \in B'_F, \forall V' \in B'_H, (\alpha^{F(U)}_{H(V)})^{-1}(H(V \cap V')) = F(U')\}$. Let $B''_F = \{\tilde{U} | \text{for all nonempty open set } \tilde{U} \subseteq U', \forall U' \in E\}$. Since B'_F is a (c)-basis, it is easy to see that $E = B''_F$. Denote $E_U = \{U' | \forall V' \in B'_H, (\alpha^{F(U)}_{H(V)})^{-1}(H(V \cap V')) = F(U')\}$. We have $\cup_{U' \in E_U} U' \subseteq U$ and $\cup_{U' \in E_U} F(U') = \cup_{V' \in B'_H} (\alpha^{F(U)}_{H(V)})^{-1}(H(V \cap V')) = (\alpha^{F(U)}_{H(V)})^{-1}(H(V)) = F(U)$. Thus $F(U) = \cup_{U' \in E_u} F(U') = F(\cup_{U' \in E_U} U')$. So $U = \cup_{U' \in E_U} U'$ by AP2 (i) b. Therefore B''_F is a (c)-basis. Given $\beta^{H(V)}_{F(U)} \alpha^{F(U)}_{H(V)} \in L$ and $U' \in E_U$, we write $\tilde{V}' = V \cap V'$, QM2 and QM3 guarantee that $\alpha^{F(U')}_{H(\tilde{V}')}$ and $\beta^{H(\tilde{V}')}_{F(U')}$ make sense. Then we have $(\beta^{H(V)}_{F(U)} \alpha^{F(U)}_{H(V)})|_{F(U')} = 1_{F(U')}$ and the two sets $L' = \{\beta^{H(\tilde{V}')}_{F(U')} \alpha^{F(U')}_{H(\tilde{V}')} | \forall U' \in B''_F\}$ and $Q = \{\tilde{V}' | \forall V \in A, \forall V' \in B'_H\}$.

Next we verify the following identity:

$$\alpha^{F(U')}_{H(\tilde{V}')} \beta^{H(\tilde{V}')}_{F(U')} = 1_{H(\tilde{V}')}, \quad \tilde{V}' \in Q. \tag{6.6}$$

Since $\beta^{H_1(V')}_{F_1(U'')}$ and $\beta^{H(\tilde{V}')}_{F(U')}$ make sense for $U'' \cap U' \neq \emptyset$, we have that $\beta^{H(\tilde{V}')}_{F_1(U'' \cap U')}$ is well-defined and there exists a commutative diagram by QM3,

$$H(\tilde{V}') \cong H_1(\tilde{V}') \overset{\beta^{H_1(\tilde{V}')}_{F_1(U'' \cap U')}}{\to} F_1(U'' \cap U') \overset{\rho^{F_1(U'' \cap U')}_{F(U')}}{\to} F(U') = H(\tilde{V}') \overset{\beta^{H(\tilde{V}')}_{F(U')}}{\to} F(U').$$

Since $\beta^{H(\tilde{V}')}_{F(U')} \alpha^{F(U')}_{H(\tilde{V}')} = 1_{F(U')}$, we have that $\rho^{F_1(U'' \cap U')}_{F(U')}$ is bijective. Then the following diagram is commutative:

$$H_1(\tilde{V}') \cong H(\tilde{V}') \overset{\beta}{\twoheadrightarrow} F(U') \overset{\rho^{-1}}{\rightarrowtail\!\!\!\twoheadrightarrow} F_1(U'' \cap U') = H_1(\tilde{V}') \overset{\beta}{\to} F_1(U'' \cap U'). \tag{6.7}$$

Thus $\beta^{H_1(\tilde{V}')}_{F_1(U'' \cap U')}$ is surjective by (6.7). For $U''' = U' \cap U''$,

$$\alpha^{F_1(U''')}_{H_1(\tilde{V}')} \beta^{H_1(\tilde{V}')}_{F_1(U''')} = (\alpha^{F_1(U'')}_{H_1(V')} \beta^{H_1(V')}_{F_1(U'')})|_{H_1(\tilde{V}')} = 1_{H_1(\tilde{V}')}. \tag{6.8}$$

Since $\beta^{H(\tilde{V}')}_{F(U')}$ is surjective and $\beta^{H(\tilde{V}')}_{F(U''')}$ makes sense, the inclusion map $i : F(U''') \hookrightarrow F(U')$ is surjective by QM3. Hence $U''' = U'$. Then the identity (6.6) follows.

Lemma 6.10 *Q is a (c)-basis.*

Proof For an open set $V'_0 \in B'_H$ and a map $\alpha^{F(U''_0)}_{H_1(V'_0)} \beta^{H_1(V'_0)}_{F(U''_0)} \in S$, we have

$$H_1(V'_0) = \cup_{U' \in B''_F, U''_0 \cap U' \neq \emptyset} (\beta^{H_1(V'_0)}_{F(U''_0)})^{-1}(F_1(U''_0 \cap U')). \tag{6.9}$$

Let $H_1(\overline{V_0}') = (\beta^{H_1(V_0')}_{F(U_0'')})^{-1}(F_1(U_0'' \cap U'))$.

$$\alpha^{F_1(U_0''\cap U')}_{H_1(\overline{V_0}')}\beta^{H_1(\overline{V_0}')}_{F_1(U_0''\cap U')} = (\alpha^{F(U_0'')}_{H_1(V_0')}\beta^{H_1(V_0')}_{F(U_0'')})|_{H_1(\overline{V_0}')} = 1_{H_1(\overline{V_0}')}.$$

So $\alpha^{F_1(U_0''\cap U')}_{H_1(\overline{V_0}')}$ is surjective since $\alpha^{F(U')}_{H(\tilde{V}')}$ makes sense for $\tilde{V}' \in Q$. By QM3, $\overline{V_0}' \cap \tilde{V}' \neq \emptyset$ and there exists a commutative diagram:

$$F_1(U_0'' \cap U') \xrightarrow{\alpha} H_1(\overline{V_0}' \cap \tilde{V}') \xhookrightarrow{\tau} H_1(\overline{V_0}') = F_1(U_0'' \cap U') \xrightarrow{\alpha} H_1(\overline{V_0}'). \tag{6.10}$$

So τ is surjective by (6.10). Hence $\overline{V_0}' = \overline{V_0}' \cap \tilde{V}'$ and $\overline{V_0}' \in Q$. Since $V_0' = \cup \overline{V_0}'$, we obtain the desired result. □

Lemma 6.11 *There is a bijective map from B_F'' to Q which preserves order $\subseteq$.*

Proof Given a map $m : B_F'' \to Q$:

$$U' \mapsto \tilde{V}', \quad \forall U' \in B_F'', \quad \text{if } F(U') = (\alpha^{F(U)}_{H(V)})^{-1}(H(V' \cap V)) \text{ and } \tilde{V}' = V' \cap V.$$

If there exists a surjective map $\alpha^{F(U')}_{H'(V'')}$, then for $\tilde{V}' \cap V'' \neq \emptyset$ the following diagram is commutative by QM3:

$$F(U') \xrightarrow{\alpha} H(\tilde{V}') = F(U') \xrightarrow{\alpha} H'(\tilde{V}' \cap V'') \xhookrightarrow{\tau} H(\tilde{V}'). \tag{6.11}$$

If $\alpha^{F(U')}_{H(\tilde{V}')}$ is surjective, so is τ. Hence $\tilde{V}' \cap V'' = \tilde{V}'$. On the other hand, the following diagram is also commutative: $F(U') \xrightarrow{\alpha} H'(V'') = F(U') \xrightarrow{\alpha} H'(\tilde{V}' \cap V'') \xhookrightarrow{\tau} H'(V'')$. We know that $\tilde{V}' \cap V'' = V''$. Hence $\tilde{V}' = V''$. Therefore the map m is well-defined.

We define a map $g : Q \to B_F''$ as follows:

$$\tilde{V}' \mapsto U', \quad \text{if } \tilde{V}' = V' \cap V \text{ with } V \in A \text{ and } V' \in B_H', \text{ and } F(U') = (\alpha^{F(U)}_{H(V)})^{-1}(H(V' \cap V)).$$

Since the map $\beta^{H(\tilde{V}')}_{F(U')}$ is surjective, a similar argument shows that the map g is also well-defined. Then it is clear that we have $mg = 1_Q$, and $gm = 1_{B_F''}$. So the map m is bijective.

Given two open sets $U', U_1' \in B_F''$ such that $U' \subseteq U_1'$. Since $\alpha^{F(U')}_{H(\tilde{V}')}$ is surjective, from the commutative diagram

$$F(U') \xrightarrow{\alpha} H(\tilde{V}' \cap \tilde{V}_1') \xhookrightarrow{\tau} H(\tilde{V}') = F(U') \xrightarrow{\alpha} H(\tilde{V}'),$$

we have $\tilde{V}' \cap \tilde{V}_1' = \tilde{V}'$, so $m(U') \subseteq m(U_1')$. Therefore m is an order-preserving map. □

Now we are in the position to prove that for each $F(U') \in \langle F, \rho \rangle_{B_F''}$ there is a unique $H(\tilde{V}') \in \langle H, \tau \rangle_Q$ such that $\alpha^{F(U')}_{H(\tilde{V}')} = 1$.

Choose an open set $H_1(\tilde{V}') \in \langle H, \tau \rangle_Q$. Since $\alpha^{F(U')}_{H_1(\tilde{V}')}$ is a C^r-diffeomorphism, there is a unique H such that the set $F(U')$ and the set $H(\tilde{V}')$ are same. Furthermore $\tau^{H_1(\tilde{V}')}_{H(\tilde{V}')} =$

$(\alpha^{F(U')}_{H_1(\tilde{V}')})^{-1}$, so $\alpha^{F(U')}_{H(\tilde{V}')} = 1$ follows from QM3. If $\alpha^{F(U')}_{H'(\tilde{V}')} = 1$, then $\tau^{H(\tilde{V}')}_{H'(\tilde{V}')} = 1$. So $H = H'$ by AP2 (i). Similarly we have that for each $H(\tilde{V}') \in \langle H, \tau\rangle_Q$ there exists a unique $F_1(U') \in \langle F, \rho\rangle_{B''_F}$ such that $\beta^{H(\tilde{V}')}_{F_1(U')} = 1$. Since $\alpha^{F(U')}_{H(\tilde{V}')}\beta^{H(\tilde{V}')}_{F(U')} = 1$, $F = F_1$. Hence for each $U' \in B''_F$ there exists a bijective map

$$b_{U'} : \{F(U')|\forall F(U') \in R_{U'}\} \to \{H(\tilde{V}')|\forall H(\tilde{V}') \in R_{\tilde{V}'}\},$$

with $b_{U'}(F(U')) = H(\tilde{V}')$.

Given $b_{U'}(F(U')) = H(\tilde{V}')$ and $b_{U'}(F'(U')) = H'(\tilde{V}')$, we get $\rho^{F(U')}_{F'(U')} = \tau^{H(\tilde{V}')}_{H'(\tilde{V}')}$ by QM3. Hence the sets $G_\tau(U')$ and $G_\tau(\tilde{V}')$ are same. Since

$$\rho^{F''(U')}_{F'(U')} \cdot \rho^{F(U')}_{\tilde{F}(U')} = \rho^{F(U')}_{F'(U')} = \tau^{H(\tilde{V}')}_{H'(\tilde{V}')} = \tau^{H''(\tilde{V}')}_{H'(\tilde{V}')} \cdot \tau^{H(\tilde{V}')}_{\tilde{H}(\tilde{V}')}, \tag{6.12}$$

the weak groups $G_\tau(U')$ and $G_\tau(\tilde{V}')$ are same. So $[G_F, f]_{B''_F} \overset{m}{=} [G_H, h]_Q$. Now the proof of Theorem 6.9 is complete. □

Theorem B $(X,\Phi) \approx (Y,\Psi)$ *if and only if there are a* (c)*-basis* B_F *of* X *and a* (c)*-basis* B_H *of* Y *such that* $[G_F, f]_{B_F} \overset{m}{=} [G_H, h]_{B_H}$, *for a bijective order-preserving map* m.

Proof If $[G_F, f]_{B_F} \overset{m}{=} [G_H, h]_{B_H}$, then for each $U \in B_F$ and $G_F(U) = G_H(m(U))$

$$\rho^{F(U)}_{F'(U)} = \tau^{H(m(U))}_{H'(m(U))}, \qquad \text{for each } \rho^{F(U)}_{F'(U)} \in G_F(U).$$

Let $\alpha^{F(U)}_{H(m(U))} = 1$ for each $F(U) \in \langle F, \rho\rangle_{B_F}$ and $\alpha^{F(U)}_{H'(m(U))} = \tau^{H(m(U))}_{H'(m(U))}$. Let $\beta^{H(m(U))}_{F(U)} = 1$ for each $H(m(U)) \in \langle H, \tau\rangle_{B_H}$ and $\beta^{H(m(U))}_{F'(U)} = \rho^{F(U)}_{F'(U)}$. Since m is bijective and order-preserving, it is easy to check that both α and β defined below are morphisms:

$$\alpha = \{\alpha^{F(U)}_{H'(m(U))}|\forall U \in B_F, \forall F, \forall H'\} : \langle F, \rho\rangle_{B_F} \to \langle H, \tau\rangle_{B_H},$$

$$\beta = \{\beta^{H(m(U))}_{F'(U)}|\forall U \in B_F, \forall H, \forall F'\} : \langle H, \tau\rangle_{B_H} \to \langle F, \rho\rangle_{B_F}.$$

Obviously, we have that

$$\alpha\beta \cong 1, \qquad \text{and} \qquad \beta\alpha \cong 1. \tag{6.13}$$

So $\langle F, \rho\rangle_{B_F} \cong \langle H, \tau\rangle_{B_H}$ in $\mathcal{AS}$. Since $\langle F, \rho\rangle_{B_F} \cong \mathbf{U}(X,\Phi)$ and $\langle H, \tau\rangle_{B_H} \cong \mathbf{U}(Y,\Psi)$, we have $(X,\Phi) \approx (Y,\Psi)$ from Corollary 5.19. The other direction is clear. □

Corollary 6.12 *There are uncountably many* (c)*-basis* B_λ *of* $\mathbf{R}^4$ *such that for any* $\lambda \neq \mu$, $[G_\lambda, f]_{B_\lambda}$ *can not be bijective order-preserving to* $[G_\mu, f]_{B_\mu}$. *I.e. there are uncountably many inequivalent* (c)*-basis of* $\mathbf{R}^4$.

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关于微分同胚的一个充分必要条件

李 维 萍[1]， 于 永 溪[2]

(1. Dept. of Math., Oklahoma State University, Stillwater, Oklahoma 74078-613, U.S.A;

2. 苏州大学数学系， 215006)

摘 要: 确定两个流形是否 C^r- 微分同胚是微分流形研究中的重要课题，本文定义了反层的概念，给出了反层范畴，由此找到两个微分流形， C^r- 微分同胚的特征刻画，于是给出了一种较以往更优的判定法.