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The Morita Characterization of Projective Free Rings *

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Abstract: In this paper, we investigate the Morita characterization of projective free rings, and show that projective free rings have Morita invariant properties.

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Recall that a ring R is said to be projective free provided that every finitely generated projective right R-module is free. In this paper, we investigate the Morita characterization of projective free rings, and show that projective free rings have Morita invariant properties. We also give a description of such rings by virtue of progenerators. As an application, we prove that connected space can be determined only by Morita equivalences.

Throughout, rings are associative with identities and modules are unital. Most modules are left modules. $\underline{P}(R)$ denotes the category of finitely generated projective right R-modules. A left R-module P is said to be a progenerator provided that it is a finitely generated projective generator. We start by the following.

Lemma 1 *Let P be a finitely generated left R-module, and ${}_P\underline{S}$ be the category of direct summands of finite direct sums of P. Then it is a category with finite products, and there exists an adjoint equivalence:*

$$\langle \mathrm{Hom}(P_{\mathrm{End}P}, -), P \bigotimes_{\mathrm{End}P} -, \psi\rangle :{}_P\, \underline{S} \rightarrow \underline{P}(\mathrm{End}P).$$

Proof Obviously, ${}_P\underline{S}$ is a category with finite product. For any $Q \in{}_P \underline{S}$, we see that $\mathrm{Hom}(P_{\mathrm{End}P}, Q) \in \underline{P}(\mathrm{End}P)$ since $\mathrm{Hom}(P_{\mathrm{End}P}, -)$ is a functor preserving the finite products. Likewise, $P \bigotimes T \in{}_P \underline{S}$ for any $T \in \underline{P}(\mathrm{End}P)$. By virtue of [1, Lemma 29.4], we complete the proof.

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Theorem 2 *Let P be a finitely generated left R-module. The following are equivalent:*

(1) $\mathrm{End}_R P$ *is a projective free ring.*

(2) Every direct summands of finite direct sums of P is a finite direct sum of P.

Proof (1) $\Rightarrow$ (2) Suppose that $Q \oplus M \cong P^m$. Then $\mathrm{Hom}(P_{\mathrm{End}P}, Q) \oplus \mathrm{Hom}(P_{\mathrm{End}P}, M) \cong \mathrm{Hom}(P_{\mathrm{End}P}, P^m) \cong (\mathrm{Hom}(P_{\mathrm{End}P}, P))^m \cong (\mathrm{End}P)^m \in \underline{F}(\mathrm{End}P)$. So we can find $n \geq 1$ such that

$$P \bigotimes_{\mathrm{End}P} \mathrm{Hom}(P_{\mathrm{End}P}, Q) \cong P \bigotimes_{\mathrm{End}P} (\mathrm{End}P)^n \cong P^n.$$

In view of Lemma 1, we know that $\mathrm{Hom}(P_{\mathrm{End}P}, -), P \otimes -$ are adjoint equivalence. Thus $Q \cong P^n$, as required.

(2) $\Rightarrow$ (1) Suppose that $N \oplus K \cong (\mathrm{End}_R P)^s$. Since $P \otimes -$ is a functor preserving the finite products, we have $P \otimes N \oplus P \otimes K \cong P \otimes (\mathrm{End}P)^s \cong P^s$. Hence $P \otimes N \in_P \underline{S}$. Thus $P \otimes N \cong P^t$ for some $t \geq 1$. Using Lemma 1, we claim that $N \cong \mathrm{Hom}(P_{\mathrm{End}P}, P \otimes N) \cong \mathrm{Hom}(P_{\mathrm{End}P}, P^t) \cong (\mathrm{End}P)^t \in \underline{F}(\mathrm{End}P)$. Therefore $\mathrm{End}_R P$ is a projective free ring.

Since endomorphism rings of strongly indecomposable left R-modules are local rings, from Theorem 2, we easily derive the following consequence.

Corollary 3 *Let P be a finitely generated strongly indecomposable left R-module. Then every direct summand of finite direct sums of P is a finite direct sum of P.*

Theorem 4 *The following are equivalent:*

(1) R is a projective free ring.

(2) $\mathrm{End}_{M_n(R)} R^{n\times 1}$ *is a projective free ring.*

Proof (1) $\Rightarrow$ (2) Let $M \in_{R^{n\times 1}} \underline{S}$. Assume that $M \oplus N \cong (R^{n\times 1})^s$ with $s \geq 1$. Then we have $R^{1\times n} \otimes M \oplus R^{1\times n} \otimes N \cong R^{1\times n} \otimes (R^{n\times 1})^s \cong (R^{1\times n} \otimes R^{n\times 1})^s$, whence $R^{1\times n} \otimes M \in \underline{P}(R)$. Thus we can find a $t \geq 0$ such that $R^{1\times n} \otimes M \cong R^t$. So $M \cong M_n(R) \otimes M \cong (R^{n\times 1} \otimes R^{1\times n}) \otimes M \cong R^{n\times 1} \otimes (R^{1\times n} \otimes M) \cong R^{n\times 1} \otimes R^t$. That is, M is a finite direct sum of $R^{n\times 1}$. By virtue of Theorem 2, we claim that every finitely generated projective $\mathrm{End}_{M_n(R)} R^{n\times 1}$-module is free, as desired.

(2) $\Rightarrow$ (1) Let $P \in \underline{P}(R)$. Assume that $P \oplus Q \cong R^m$ with $m \geq 0$. Then we have $R^{n\times 1} \otimes P \oplus R^{n\times 1} \otimes Q \cong (R^{n\times 1})^m$, and then $(R^{n\times 1}) \otimes P \in_{R^{n\times 1}} \underline{S}$. Since $\mathrm{End}_{M_n(R)} R^{n\times 1}$ is a projective free ring, from Theorem 2, we have $r \geq 0$ such that $R^{n\times 1} \otimes P \cong (R^{n\times 1})^r$. Hence $P \cong R \otimes P \cong (R^{1\times n} \otimes R^{n\times 1}) \otimes P \cong R^{1\times n} \otimes (R^{n\times 1})^t \cong (R^{1\times n} \otimes R^{n\times 1})^r \cong R^r \in \underline{F}(R)$. Consequently, R is a projective free ring.

Recall that R is said to be a UCP ring provided that every finitely generated stable free R-module is free. As an immediate consequence, we can derive the following fact.

Corollary 5 *The following are equivalent:*

(1) R is a projective free ring.

(2) For any $Q \in \underline{P}(M_n(R))$, there there exists some $m \geq 1$ such that $Q \cong (R^{n\times 1})^m$.

(3) $\psi : K_0 M_n(R) \cong \mathbf{Z}, \psi([R^{n\times 1}]) = 1$ and $M_n(R)$ is a UCP ring.

Recall that two rings S and R are Morita equivalent, abbreviated $S \approx R$ in case there are additive equivalences between the categories of left S-modules and left R-modules. Now we describe projective free properties by virtue of Morita equivalences.

Theorem 6 *The following are equivalent:*

(1) R is a projective free ring.

(2) For any $S \approx R$, there exists a progenerator ${}_SP$ such that $\mathrm{End}_S P$ *ia a projective free ring.*

Proof (2) $\Rightarrow$ (1) Since $R \approx R$, there exists a progenerator ${}_RP$ such that $\mathrm{End}_R P$ is a projective free ring. As ${}_RP$ is a progenerator, we know that $R \in_P \underline{S}$. On the other hand, $\mathrm{End}_R P$ is a projective free ring, by Theorem 2, there is some $t \geq 1$ such that $R \cong P^t$. Let $Q \in \underline{P}(R)$. Then $P \bigotimes Q \in_P \underline{S}$, whence $P \bigotimes Q \cong P^m$. So we see that $P^t \bigotimes Q \cong P^{tm} \cong R^m$, and then $Q \cong R \bigotimes Q \cong P^t \bigotimes Q \cong R^m \in \underline{F}(R)$, as asserted.

(1) $\Rightarrow$ (2) For any $S \approx R$, there exists a progenerator $Q \in \underline{P}(R)$ such that $S \cong \mathrm{End}_R Q$ from [1, Corollary 22.4]. Since R is a projective free ring, we can find some $n \geq 1$ such that $Q \cong R^n$, whence $S \cong \mathrm{End} R^n \cong M_n(R)$. Thus there is an isomorphism $\psi : M_n(R) \cong S$. Now we construct a map $\phi : \mathrm{End}_{M_n(R)} R^{n\times 1} \to \mathrm{End}_S(S \bigotimes R^{n\times 1})$ given by $\phi(\delta) = 1 \bigotimes \delta$ for any $\delta \in \mathrm{End}_{M_n(R)} R^{n\times 1}$. Clearly, ϕ is a monomorphism. For any $f \in \mathrm{End}_S(S \bigotimes R^{n\times 1})$, we have $1 \bigotimes f : R^{n\times 1} \to R^{n\times 1}$. It is easy to check that $\phi(1 \bigotimes f) = 1 \bigotimes (1 \bigotimes f) = f$. So ϕ is a ring isomorphism. Set $P = S \bigotimes R^{n\times 1}$. Then we see that $S \cong S \bigotimes M_n(R) \cong S \bigotimes (R^{n\times 1})^n \cong P^n$, hence ${}_SP$ is a progenerator of S-module category. In view of Theorem 4, we complete the proof.

Corollary 7 *Let R be a local ring or a principal ideal domain. Then for any $S \approx R$, there exists a progenerator ${}_SP$ such that* $\mathrm{End}_S P$ *is a projective free ring.*

Proof Since R is a local ring or a principal ideal domain, it must be a projective free ring. By virtue of Theorem 6, we complete the proof.

It is well known that $K_0(R)$ can be seen as a commutative ring when R is commutative, where $[P][Q] = [P \bigotimes Q]$. A ring R is said to be connected provided that it has only trivial idempotents. Now we investigate connected rings by virtue of Grothendieck rings as follows.

Corollary 8 *Let R be a commutative ring. Then the following hold:*

(1) R is a connected ring.

(2) For any $S \approx K_0 R$, there exists a progenerator ${}_SP$ such that $\mathrm{End}_S P$ *is a projective free ring.*

Proof (1) $\Rightarrow$ (2) Since R is a connected ring, by virtue of [2, Theorem 1.1], we know that $K_0(R)$ is a projective free ring. In view of Theorem 6, we claim that for any $S \approx K_0 R$, there exists a progenerator ${}_SP$ such that $End_S P$ is a projective free ring.

(2) $\Rightarrow$ (1) According to Theorem 6, we know that $K_0(R)$ is a projective free ring. Moreover, it is a connected ring. Given $P \in \underline{P}(R)$ with $P \bigotimes P \cong P$. Then $[P]^2 = [P]$ holds in $K_0(R)$. So we have $[P] = 0$ or $[P] = [R]$. If $[P] = 0$, then $P = 0$. If $[P] = [R]$, then there exists some $m \geq 1$ such that $P \oplus R^m \cong R^{m+1}$. Hence $P \cong \det P \cong \det P \bigotimes \det R^m \cong \det(P \oplus R^m) \cong \det R^{m+1} \cong R$. From [2, Corollary 1.2], we conclude that R is a connected ring.

If X is a topological space and $C(X)$ is the ring of all continuous real-valued functions on X, then we observe the following fact.

Corollary 9 *Let X be a topological space. Then the following hold:*

(1) X is a connected space.

(2) For any $S \approx K_0C(X)$, there exists a progenerator ${}_SP$ such that End_SP is a projective free ring.

Proof Since X is a connected space if and only if $C(X)$ is a connected ring. Thus, the result follows from Corollary 8.

Similarly to the proof in Corollary 9, we can derive the following.

Proposition 10 *Let R be a commutative ring. Then the following hold:*

(1) Every finitely generated projective R-module is stably free.

(2) For any $S \approx R$, there exists a progenerator ${}_SP \in \underline{P}(S)$ such that $\psi : K_0S \cong \mathbf{Z}$ with $\psi([P]) = 1$.

Corollary 11 *Let X be a topological space. Then the following hold:*

(1) X is a connected space.

(2) For any $S \approx K_0C(X)$, there exists a progenerator ${}_SP \in \underline{P}(S)$ such that $\psi : K_0S \cong \mathbf{Z}$ with $\psi([P]) = 1$.

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投射自由环的 Morita 特征

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摘 要: 本文研究投射自由环的 Morita 特征，证明了投射自由环具有 Morita 不变性.