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On the Zero-One-Pole Set of a Meromorphic Function *

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Let $\{a_n\},\{b_n\}$ and $\{p_n\}$ be three disjoint sequences with no finite limit points. If it is possible to construct a meromorphic function N in the plane whose zeros, one points and poles are exactly $\{a_n\},\{b_n\}$ and $\{p_n\}$ respectively, where their multiplicities are taken into consideration, then the given triple $(\{a_n\},\{b_n\},\{p_n\})$ is called the zero-one-pole set (of N). In general an arbitrary triad $(\{a_n\},\{b_n\},\{p_n\})$ is not a zero-one-pole set of any meromorphic function. This was proved by Rubel and Yang[6] explicitly for entire functions. Ozawa[5] proved the following.

Theorem A *Let $\{a_n\}$ and $\{b_n\}$ be two arbitrary disjoint infinite sequences with no finite limit points. Let b_1 be different from b_2. Then one of the following three pairs*

$$(\{a_n\},\{b_n\}_{n=1}^{\infty}),(\{a_n\},\{b_n\}_{n=2}^{\infty}),(\{a_n\},\{b_n\}_{n=3}^{\infty}\cup\{b_1\})$$

is not a zero-one set of any entire function.

An example in [5] shows that two pairs are really zero-one sets in Theorem A. As the first supplement Ozawa also proved that if one of $\{a_n\}$ and $\{b_n\}$ is a nonempty finite sequence then three pairs can be reduced to two pairs in Theorem A.

We proved the following

Theorem 1 *Let $\{a_n\},\{b_n\}$ and $\{p_n\}$ be three disjoint sequences with no finite limit points. Let $\{c_n\}$ and $\{d_n\}$ be two nonempty distinct subsequences of $\{b_n\}$ with*

$$\sum_{c_n\neq 0}|c_n|^{-1}+\sum_{d_n\neq 0}|d_n|^{-1}<\infty.$$

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Then at least one of the following three triads

$$(\{a_n\},\{b_n\},\{p_n\}),(\{a_n\},\{b_n\}\backslash\{c_n\},\{p_n\}),(\{a_n\},\{b_n\}\backslash\{d_n\},\{p_n\})$$

is not a zero-one-pole set of any meromorphic function.

Theorem 2 *Let $\{a_n\},\{b_n\}$ and $\{p_n\}$ be three disjoint sequences with no finite limit points and*

$$\sum_{b_n\neq 0}|b_n|^{-1}+\sum_{p_n\neq 0}|p_n|^{-1}<\infty. \tag{1}$$

Then at least one of the following two triads

$$(\{a_n\},\{b_n\},\{p_n\}),(\{a_n\},\{b_n\}\backslash\{c_n\},\{p_n\})$$

is not a zero-one-pole set of any meromorphic function, for an arbitrary nonempty subsequence $\{c_n\}$ of $\{b_n\}$ with $\{c_n\}\neq\{b_n\}$.

Their proofs depend on the impossibility of Borel's identity.

Lemma *Let $P_0,P_1,\cdots,P_n(P_j\not\equiv 0, 0\le j\le n, n\ge 1)$ be entire functions satisfying $m(r,P_j)=o(r)(r\to\infty)(j=0,\cdots,n)$ and let $g_1,g_2,\cdots,g_n$ be nonconstant entire functions. Then an identity of the following form $\sum_{j=1}^{n}P_je^{g_j}=P_0$ is impossible.*

This had been stated in several ways (see [2], [3] and [7]). Moreover this also is an easy consequence of Lemma 1 in [4].

Remark If the condition (1) in Theorem 2 is replaced by

$$\sum_{a_n\neq 0}|a_n|^{-1}+\sum_{c_n\neq 0}|c_n|^{-1}+\sum_{p_n\neq 0}|p_n|^{-1}<\infty,$$

then the conclusion is still true.

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