Journal of Mathematical Research & Exposition
Vol.20, No.2, 166-170, May 2000

Subsets with Finite Measure of Multifractal Hausdorff Measures *

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Abstract: Let μ be a Borel Probability measure on R^d. $q, t, \in R$. Let $\mathcal{H}_\mu^{q,t}$ denote the multifractal Hausdorff measure. We prove that, when μ satisfies the so-called Federer condition, for a closed subset $E \in R^n$, such that $\mathcal{H}_\mu^{q,t}(E) > 0$, there exists a compact subset F of E with $0 < \mathcal{H}_\mu^{q,t}(F) < \infty$, i.e., the finite measure subsets of multifractal Hausdorff measure exist.

Key words: multifractal Hausdorff measure; finite measure subset; net measure.

Classification: AMS(1991) 28A12/CLC O174.12

Document code: A **Article ID:** 1000-341X(2000)02-0166-05

1. Introduction

In recent years, many theoretical physicists and mathematicans have studied the so-called multifractal theory (see [1],[2],[3]). A number of claims have been made on the basis of the heuristics and physical intuition. Recently, Olsen [4] developed a mathematical vigorous multifractal formalism based on a natural multifractal generalization of the centred Hausdorff measure and the packing measure, and discussed the relations between the multifractal dimension and the spectra function, and many properties of the multifractal measure and dimension are obtained. The existence of finite measure subsets is an important and useful property of Hausdorff measure, and it is the essential work of Besconvitch(1952)(See [5]). In this paper, we discuss the finite measure subsets of multifractal Hausdorff measure.

2. Prelimilary and Main results

Our analysis is based on the multifractal formalism introduced by Olsen in [4].

Let X be a metric space, $\mathcal{P}(X)$ be the family of Borel probability measure on X.

A countable family $\mathcal{B} = (B(x_i, r_i))_i$ of closed balls of X is called a centred δ-covering of E if $E \subset \cup_i B(x_i, r_i), x_i \in E$, and $0 < r_i < \delta$ for all i.

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*Received date: 1997-07-18
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Let $E \subset X, t \geq 0$, denote the t-dimensional Hausdorff measure and Hausdorff dimension of E by $\mathcal{H}^t(E)$ and $\dim E$(see [6] for the definitions)

We now define the multifractal Hausdorff measure $\mathcal{H}_\mu^{q,t}$.

For $\mu \in \mathcal{P}, E \subset X, q, t \in R$, and $\delta > 0$, write

$$\begin{aligned}
\overline{\mathcal{H}}_{\mu,\delta}^{q,t}(E) &= \inf\{\sum_i (\mu(B(x_i, r_i)))^q (2r_i)^t : (B(x_i, r_i))_i \quad \text{is} \\
&\qquad \text{a centred } \delta\text{-covering of } E\}, \quad E \neq \emptyset; \\
\overline{\mathcal{H}}_{\mu,\delta}^{q,t}(\emptyset) &= 0; \\
\overline{\mathcal{H}}_{\mu}^{q,t}(E) &= \sup_{\delta>0} \overline{\mathcal{H}}_{\mu,\delta}^{q,t}(E); \\
\mathcal{H}_{\mu}^{q,t}(E) &= \sup_{F \subset E} \overline{\mathcal{H}}_{\mu}^{q,t}(F).
\end{aligned}$$

We set $0^0 = 1$ and $0^q = \infty$ if $q < 0$.

$\mathcal{H}_\mu^{q,t}$ is a metric outer measure, and it is a measure on the Borel algebra ([4]).By proposition 1.1 of Olsen [4],we can define the multifractal Hausdorff dimension $\dim_\mu^q(E)$ of E as follows: $\dim_\mu^q(E) = \sup\{t : \ \mathcal{H}_\mu^{q,t}(E) = \infty\} = \inf\{t : \ \mathcal{H}_\mu^{q,t}(E) = 0\}$.

It can be easily seen that $\dim(E) = \dim_\mu^0(E)$, and $\dim_\mu^q$ is a monotonous and σ-stable index.

For $\mu \in \mathcal{P}(X)$ and $a > 1$ write

$$T_a(\mu) = \limsup_{r\to 0}\Big(\sup_{x\in \operatorname{supp}\mu} \frac{\mu(B(x, ar))}{\mu(B(x, r))}\Big).$$

Let $\mathcal{P}_F(X) = \{\mu : T_a(\mu) < \infty$ for some $a > 1\}$. $\mu \in \mathcal{P}_F(X)$ is called a Federer measure.

Lemma 2.1 *If $\mu \in \mathcal{P}_F(R^d), a > 1$,then there exist constants $r_0, C > 0$, such that*

$$C^{-1} \leq \frac{\mu(B(x, ar))}{\mu(B(x, r))} \leq C \tag{2.1}$$

for all $x \in \operatorname{supp}\mu$, and $r < r_0$.

It can be easily proved by the definition of $\mathcal{P}_F(R^d)$.

Theorem 2.2 *Let E be a closed subset of R^d, $\mu \in \mathcal{P}_F(R^d), \mathcal{H}_\mu^{q,t}(E) = \infty$, Let $A > 0$, there exists a compact subset F of E such that $0 < A \leq \mathcal{H}_\mu^{q,t}(F) < \infty$.*

3. Proof of Theorem 2.2

To prove Theorem 2.2, we introduce a net measure $\mathcal{M}_\mu^{q,t}$, which was introduced by Olsen in [8]. For $n \in N$, write

$$\mathcal{U}_n = \{\prod_{i=1}^d [\frac{l_i}{2^n}, \frac{l_i + 1}{2^n}) : l_1, l_2, \cdots l_d \in Z\}.$$

The family $\mathcal{U}_n$ is the class of half-dyadic cubes of order n. For $u \in \mathcal{U}_n$, let

$$[u] = \sum_{v \in \mathcal{U}_n d(u,v)=0} v,$$

where $d(u,v) = \inf\{d(x,y) : x \in u, y \in v\}$, i.e. $[u]$ is the union of u and its immediatly adjacent neighbouring dyadic cubes of order n. For $\mu \in \mathcal{P}(R^d), E \in R^d, q,t \in R$, and $n \in N$,write

$$\mathcal{M}^{q,t}_{\mu,n}(E) = \inf\{\sum_i (\mu([u_i]))^q (2^{-n_i})^t : E \subset \sum_i u_i, u_i \in \mathcal{U}_{n_i}, n_i \geq n\}, E \neq \emptyset;$$
$$\mathcal{M}^{q,t}_{\mu,n}(\emptyset) = 0;$$
$$\mathcal{M}^{q,t}_{\mu}(E) = \sup_n \mathcal{M}^{q,t}_{\mu,n}(E).$$

Lemma 3.1 (Olsen[8] lemma 5.2.1) *Let $\mu \in \mathcal{P}(R^d), q,t \in R$, and $n \in N$.*

(i) $\mathcal{M}^{q,t}_{\mu}$ is a regular metric outer measure, in particular $\lim_{i\to\infty} \mathcal{M}^{q,t}_{\mu}(E_i) = \mathcal{M}^{q,t}_{\mu}(\lim_{i\to\infty} E_i)$ for any increasing sequence $E_i \to E$;

(ii) $\lim_{i\to\infty} \mathcal{M}^{q,t}_{\mu,n}(E_i) = \mathcal{M}^{q,t}_{\mu,n}(\lim_{i\to\infty} E_i)$ for any increasing sequence $E_i \to E$. (3.1)

Lemma 3.2 (Olsen [8] lemma 5.2.2) *Let $\mu \in \mathcal{P}(R^d)$, $q,t \in R, E \in R^d$ there exists a constant C_1 such that*

$$C_1{}^{-1}\mathcal{M}^{q,t}_{\mu} \leq \mathcal{H}^{q,t}_{\mu} \leq C_1\mathcal{M}^{q,t}_{\mu}. \tag{3.2}$$

Lemma 3.3 *Let $\mu \in \mathcal{P}(R^d)$, $q,t \in R, E \in R^d$ there exists a constant C_2 such that*

$$\mathcal{M}^{q,t}_{\mu,n}(E) \leq C_2\overline{\mathcal{H}}^{q,t}_{\mu,\delta}(E) \text{ for } 2^{-(n+2)} \leq \delta < 2^{-(n+1)}. \tag{3.3}$$

It can be easily obtained from the proof of Lemma 5.2.2 in [8].

Lemma 3.4 *Let μ be a Federer probability measure,$\{E_i\}$ be a decreasing sequence of compact subsets of R^d, then for sufficent small $\delta > 0$ there exists a constant C_3 such that*

$$2^t C_3\overline{\mathcal{H}}^{q,t}_{\mu,\delta}(\lim_{i\to\infty} E_i) \geq \lim_{i\to\infty} \overline{\mathcal{H}}^{q,t}_{\mu,2\delta}(E_i). \tag{3.4}$$

Proof Since μ is a Federer measure, by lemma 1.1 for sufficently small δ, there exists a constant $C_3 > 0$ such that

$$C_3{}^{-1} \leq \frac{\mu(B(x,2r))^q}{\mu(B(x,r))^q} \leq C_3$$

for $x \in \mathrm{supp}\mu$ and $0 < r < \delta$.

Suppose $(B(x_i,r_i))_i$ is a centred δ-covering of $\lim_{i\to\infty} E_i$, let $V = \sum_i B^o(x_i,2r_i)$ (For this section we use $B^o(x,r)$ to denote the open ball, and $B(x,r)$ to denote the closed ball). Then there exists j such that $E_j \subset V$. Otherwise $\{E_j \setminus v\}$ is a decreasing sequense of non-empty compact sets, so its limit sets $(\lim E_i \setminus v) \neq \emptyset$, contradicting with $\lim E_i \subset \sum_i B^o(x_i,2r_i)$. So we have $E_j \subset \cup_i B(x_i,2r_i)$. Since

$$2^t C_3 \sum_i (2r_i)^t \mu(B(x_i,r_i))^q \geq \sum_i (4r_i)^t \mu(B(x_i,2r_i))^q \geq \overline{\mathcal{H}}^{q,t}_{\mu,2\delta}(E_j) \geq \lim_{j\to\infty} \overline{\mathcal{H}}^{q,t}_{\mu,2\delta}(E_j),$$

we have $2^t C_3\overline{\mathcal{H}}^{q,t}_{\mu,\delta}(\lim_{i\to\infty} E_i) \geq \lim_{i\to\infty} \overline{\mathcal{H}}^{q,t}_{\mu,2\delta}(E_i)$.

Now we prove Theorem 2.2.

Proof We prove our theorem for $d = 1$. For higher dimensions the proof is analogous.

Since $\mathcal{M}_\mu^{q,t}(E) = \infty$, We can find an integer m such that

$$2^t C_2 C_3 A < \mathcal{M}_{\mu,m}^{q,t}(E) < \infty,$$

where C_2, C_3 are the constants in Lemma 3.3 and Lemma 3.4.

We inductively define a decreasing sequence $\{E_k\}$ of subset of E as follows:

Let $E_m = E$ for $n \geq m$.

Suppose $I \subset \mathcal{U}_n$.

(1) If $\mathcal{M}_{\mu,n+1}^{q,t}(E_n \cap I) \leq 2^{-tn}\mu([I])^q$ let $E_{n+1} \cap I = E_n \cap I$.

When using I as a covering interval in calculating $\mathcal{M}_{\mu,n}^{q,t}(E)$, we have

$$\mathcal{M}_{\mu,n}^{q,t}(E_n \cap I) = \min\{\mathcal{M}_{\mu,n+1}^{q,t}(E_n \cap I), 2^{-tn}\mu([I])^q\}.$$

So

$$\mathcal{M}_{\mu,n+1}^{q,t}(E_{n+1} \cap I) = \mathcal{M}_{\mu,n}^{q,t}(E_n \cap I) \leq 2^{-tn}\mu([I]))^q. \tag{3.5}$$

(2) If $\mathcal{M}_{\mu,n+1}^{q,t}(E_n \cap I) > 2^{-tn}\mu([I]))^q$ let $E_{n+1} \cap I$ be a subset of $E_n \cap I$ satisfied $\mathcal{M}_{\mu,n+1}^{q,t}(E_{n+1} \cap I) = 2^{-tn}\mu([I])^q$ (since $\mathcal{M}_{\mu,n+1}^{q,t}(E_n \cap I \cap [-\infty, x))$ is finite and continuous in x, such a subset exists) so we have

$$\mathcal{M}_{\mu,n}^{q,t}(E_n \cap I) = \min\{2^{-tn}\mu([I])^q, \mathcal{M}_{\mu,n+1}^{q,t}(E_n \cap I)\} = 2^{-tn}\mu([I])^q.$$

Conbine (1),(2) we always have

$$\mathcal{M}_{\mu,n+1}^{q,t}(E_{n+1} \cap I) = \mathcal{M}_{\mu,n+1}^{q,t}(E_n \cap I) \leq 2^{-tn}\mu([I])^q \quad n \geq m. \tag{3.6}$$

Sum (3.6) over all $I \in \mathcal{U}_n$, we have

$$\mathcal{M}_{\mu,n+1}^{q,t}(E_{n+1}) = \mathcal{M}_{\mu,n}^{q,t}(E_n) \quad n \geq m.$$

Iterate upper procedure, we have

$$\mathcal{M}_{\mu,n}^{q,t}(E_n) = \mathcal{M}_{\mu,m}^{q,t}(E_m) \quad n \geq m. \tag{3.7}$$

For $I \in \mathcal{U}_n$, if $m \leq n < l$ then $E_l \subset E_{n+1}$. By (3.6) we have

$$\mathcal{M}_{\mu,n+1}^{q,t}(E_l \cap I) \leq \mathcal{M}_{\mu,n+1}^{q,t}(E_{n+1} \cap I) \leq 2^{-tn}\mu([I])^q. \tag{3.8}$$

So

$$\mathcal{M}_{\mu,n}^{q,t}(E_l \cap I) = \min\{\mathcal{M}_{\mu,n+1}^{q,t}(E_l \cap I), 2^{-tn}\mu([I])^q\} = \mathcal{M}_{\mu,n+1}^{q,t}(E_l \cap I). \tag{3.9}$$

Sum (3.9) for all $I \in \mathcal{U}_n$ we have

$$\mathcal{M}_{\mu,n+1}^{q,t}(E_l) = \mathcal{M}_{\mu,n}^{q,t}(E_l) \quad m \leq n < l. \tag{3.10}$$

By (3.7) and (3.10) we have

$$\mathcal{M}_{\mu,m}^{q,t}(E_l) = \mathcal{M}_{\mu,l}^{q,t}(E_l) = \mathcal{M}_{\mu,m}^{q,t}(E_m) \quad l \geq m. \tag{3.11}$$

Let $F = \cap_{n=1}^{\infty} E_n$. By Lemma 2.2 and (3.7)

$$\begin{aligned} &\mathcal{H}_{\mu}^{q,t}(F) \le C_1 \mathcal{M}_{\mu}^{q,t}(F) = C_1 \lim_{n\to\infty} \mathcal{M}_{\mu,n}^{q,t}(F) \\ &\le C_1 \liminf_{n\to\infty} \mathcal{M}_{\mu,n}^{q,t}(E_n) = C_1 \mathcal{M}_{\mu,m}^{q,t}(E_m) < \infty \end{aligned} \tag{3.12}$$

On the other hand by Lemma 2.4 and Lemma 2.3

$$\begin{aligned} 2^t C_3 \mathcal{H}_{\mu}^{q,t}(F) &\ge 2^t C_3 \overline{\mathcal{H}}_{\mu,2^{-(m+4)}}^{q,t}(F) \ge \lim_{i\to\infty} \overline{\mathcal{H}}_{\mu,2^{-(m+3)}}^{q,t}(E_i) \ge C_2{}^{-1} \lim_{i\to\infty} \mathcal{M}_{\mu,m+1}^{q,t}(E_i) \\ &= C_2{}^{-1} \mathcal{M}_{\mu,m+1}^{q,t}(E_{m+1}) \ge C_2{}^{-1} 2^t C_2 C_3 A = 2^t C_3 A \end{aligned} \tag{3.13}$$

Combine (2.12) and (2.13) we have $0 < \mathcal{H}_{\mu}^{q,t}(F) < \infty$.

Acknowlegement The authors thank Prof. Hu Dihe and Prof. Liu Luqin for their suggestion and encouragement.

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重分形 Hausdorff 测度的有限测度子集

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摘 要: 设 μ 为 R^d 上 Borel 概率测度，$q, t \in R$, 记 $\mathcal{H}_{\mu}^{q,t}$ 为 Olsen[4] 定义的重分形 Hausdorff 测度，证明了当 μ 为测度时， $\mathcal{H}_{\mu}^{q,t}$ 的有限测度子集存在.