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On Some Criteria for Close-To-Convexity of Meromorphic Functions *

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Abstract: By using the Ruscheweyh type derivative for meromorphic functions and some properties of the classes C_n studied earlier by Sarangi and Suguna Uralegaddi[5], some new criteria for close-to-convexity of meromorphic functions are obtained.

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1. Introduction

Let Σ denote the class of functions of the form $f(z) = \frac{1}{z} + \sum_{n=0}^{\infty} a_n z^n$ that are regular in the punctured disk $E = \{z : 0 < |z| < 1\}$ with a simple pole at $z = 0$. The Hadamard product or convolution of $f, g \in \Sigma$ will be denoted by $f * g$.

Let

$$\begin{aligned} D^n f(z) &= \frac{1}{z(1-z)^{n+1}} * f(z), \quad n \in N_0 = \{0, 1, 2, \cdots\} \\ &= \frac{1}{z} \frac{(z^{n+1} f(z))^{(n)}}{n!} \\ &= \frac{1}{z} + (n+1)a_0 + \frac{(n+1)(n+2)}{2!} a_1 z + \cdots. \end{aligned}$$

The symbol $D^n f$ which is referred as the n^{th} order Ruscheweyh type derivative of $f \in \Sigma$ was introduced by Ganigi and Uralegaddi in [1]. In [4] Sarangi and Suguna Uralegaddi have proved that if $f \in \Sigma$ satisfies the condition $\text{Re}\{1 + zf''(z)/f'(z)\} > -3/2$ then f is meromorphically close-to-convex of order $1/2$. Using this result in [5] they have also shown that functions in C_n are meromorphically close-to-convex of order $1/2$ through basic

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inclusion relationship $C_{n+1} \subset C_n$, $n \in N_0$, where C_n is the class of functions in Σ that satisfies the condition.

$$\mathrm{Re}\{\frac{(D^{n+1}f(z))'}{(D^n f(z))'}\} > \frac{2n+1}{2n+2}, \quad z \in U = \{z : |z| < 1\}. \tag{1}$$

In this paper we obtain some criteria for close-to-convexity of $f \in \Sigma$ by using the Ruscheweyh type derivative for $f \in \Sigma$ and the properties of the class C_n. Methods used are similar to those of M.Obradovic[3]. We need the following lemma due to Jack[2].

Lemma *Let w be nonconstant and analytic in the unit disk $U, w(0) = 0$. If $|w|$ attains its maximum value on the circle $|z| = r < 1$ at z_0, we have $z_0 w'(z_0) = kw(z_0)$ where k is real number and $k \geq 1$.*

2. Some criteria for close-to-convexity

Theorem 1 *Let $f \in \Sigma, \frac{\alpha}{2n+4} \geq \frac{\beta}{2n+1} \geq 0, n \in N_0$ and let*

$$\mathrm{Re}\{\alpha\frac{(D^{n+2}f(z))'}{(D^{n+1}f(z))'} + \beta\frac{(D^n f(z))'}{(D^{n+1}f(z))'}\} > \alpha(\frac{2n+3}{2n+4}) + \beta(\frac{2n+2}{2n+1}), \quad z \in U. \tag{2}$$

then $f \in C_n$. Hence f is meromorphic close-to-convex of order $1/2$.

Proof Let $f \in \Sigma$ satisfy the condition (2). Now we shall show that $f \in C_n$, i.e.

$$\mathrm{Re}\{\frac{(D^{n+1}f(z))'}{(D^n f(z))'}\} > \frac{2n+1}{2n+2}, \quad z \in U.$$

Define $w(z)$ in U by

$$\{\frac{(D^{n+1}f(z))'}{(D^n f(z))'}\} = \frac{2n+1}{2n+2} + \frac{1}{2n+2} \cdot \frac{1-w(z)}{1+w(z)} = \frac{(n+1)+nw(z)}{(n+1)(1+w(z))} \tag{3}$$

Clearly $w(z)$ is analytic in U and $w(0) = 0$. We shall prove that $|w(z)| < 1$ in U. Differentiating (3) logarithmically and using the identity

$$z(D^n f(z))'' = (n+1)(D^{n+1}f(z))' - (n+3)(D^n f(z))', \tag{4}$$

which follows from the identity [1]

$$z(D^n f(z))' = (n+1)D^{n+1}f(z) - (n+2)D^n f(z). \tag{5}$$

We have

$$\begin{aligned}\frac{(D^{n+2}f(z))'}{(D^{n+1}f(z))'} &= \frac{2n+3}{2n+4} + \frac{n+1}{(n+2)(2n+2)} \cdot \frac{1-w(z)}{1+w(z)} - \\ &\quad \frac{zw'(z)}{(n+2)(1+w(z))(n+1+nw(z))}.\end{aligned} \tag{6}$$

Hence we have

$$\alpha\frac{(D^{n+2}f(z))'}{(D^{n+1}f(z))'}+\beta\frac{(D^{n}f(z))'}{(D^{n+1}f(z))'}$$
$$=\frac{\alpha}{n+2}[\frac{2n+3}{2}+\frac{n+1}{2n+2}\cdot\frac{1-w(z)}{1+w(z)}-\frac{zw'(z)}{(1+w(z))(n+1+nw(z))}]+$$
$$\beta\frac{(n+1)(1+w(z))}{(n+1)+nw(z)}. \tag{7}$$

Now we claim that $|w(z)|<1$. For otherwise by Jack's lemma there exists $z_0, |z_0|<1$ such that $|w(z_0)|=1$,

$$z_0w'(z_0)=kw(z_0),\quad k\geq 1.$$

Then from (7), we have

$$\alpha\frac{(D^{n+2}f(z_0))'}{(D^{n+1}f(z_0))'}+\beta\frac{(D^{n}f(z_0))'}{(D^{n+1}f(z_0))'}$$
$$=\frac{\alpha}{n+2}[\frac{2n+3}{2}+\frac{n+1}{2n+2}\cdot\frac{1-w(z_0)}{1+w(z_0)}-\frac{kw(z_0)}{(1+w(z_0))(n+1+nw(z_0))}]+$$
$$\beta\frac{(n+1)(1+w(z_0))}{(n+1)+nw(z_0)}. \tag{8}$$

Thus we have

$$\mathrm{Re}\{\alpha\frac{(D^{n+2}f(z_0))'}{(D^{n+1}f(z_0))'}+\beta\frac{(D^{n}f(z_0))'}{(D^{n+1}f(z_0))'}$$
$$\leq\alpha\frac{4n^2+8n+2}{(2n+4)(2n+1)}+\beta\frac{2n+2}{2n+1}<\alpha\frac{2n+3}{2n+4}+\beta\frac{2n+2}{2n+1}.$$

Which contradicts (2). It follows that $f\in C_n$. Hence f is close-to-convex of order $1/2$. Since for $z=0$ the left hand side of (2) have the value $\alpha+\beta$, the condition $\frac{\alpha}{2n+4}\geq\frac{\beta}{2n+1}\geq 0$ is necessary.

Theorem 2 *Let $f\in\Sigma, \alpha\geq 0, \beta\geq 0, n\in N_0$. If*

$$|\frac{(D^{n+2}f(z))'}{(D^{n+1}f(z))'}-1|^{\alpha}|\frac{(D^{n+1}f(z))'}{(D^{n}f(z))'}-1|^{\beta}<(\frac{n+1}{(n+2)(2n+1)})^{\alpha}2^{-\beta}(n+1)^{-\beta}, \tag{9}$$

then $f\in C_n$. Hence f is meromorphic close-to-convex of order $1/2$.

Proof Let $f\in\Sigma$ satisfy the inequality (9). Proceeding as in Theorem 1, from (3) and (6) we have

$$|\frac{(D^{n+2}f(z))'}{(D^{n+1}f(z))'}-1|^{\alpha}|\frac{(D^{n+1}f(z))'}{(D^{n}f(z))'}-1|^{\beta}$$
$$=(n+2)^{-\alpha}(n+1)^{-\beta}|\frac{1}{2}-\frac{n+1}{2n+2}\cdot\frac{1-w(z)}{1+w(z)}+$$
$$\frac{zw'(z)}{(1+w(z))(n+1+nw(z))}|^{\alpha}|\frac{w(z)}{1+w(z)}|^{\beta}. \tag{10}$$

Now we claim that $|w(z)| < 1$. For otherwise by Jack's lemma there is a $z_0, |z_0| < 1$ such that $|w(z_0)| = 1, z_0 w'(z_0) = kw(z_0), k \geq 1$. Then from (10) we have

$$|\frac{(D^{n+2}f(z_0))'}{(D^{n+1}f(z_0))'} - 1|^{\alpha}|\frac{(D^{n+1}f(z_0))'}{(D^n f(z_0))'} - 1|^{\beta}$$

$$= (n+2)^{-\alpha}(n+1)^{-\beta}|\frac{1}{2} - \frac{n+1}{2n+2}\cdot\frac{1-w(z_0)}{1+w(z_0)} + \frac{kw(z_0)}{(1+w(z_0))(n+1+nw(z_0))}|^{\alpha}|\frac{w(z_0)}{1+w(z_0)}|^{\beta}. \tag{11}$$

$$\mathrm{Re}\{\frac{1}{2} - \frac{n+1}{2n+2}\cdot\frac{1-w(z_0)}{1+w(z_0)} + \frac{kw(z_0)}{(1+w(z_0))(n+1+nw(z_0))}\} \tag{12}$$

$$\geq \frac{1}{2} + \frac{1}{2(2n+1)} = \frac{n+1}{2n+1}, \mathrm{Re}\{\frac{w(z_0)}{1+w(z_0)}\} = \frac{1}{2}. \tag{13}$$

Since $|z| \geq |\mathrm{Re} z|$ for all z, we have from (11)

$$|\frac{(D^{n+2}f(z_0))'}{(D^{n+1}f(z_0))'} - 1|^{\alpha}|\frac{(D^{n+1}f(z_0))'}{(D^n f(z_0))'} - 1|^{\beta}$$

$$\geq (n+2)^{-\alpha}(n+1)^{-\beta}(\frac{n+1}{2n+1})^{\alpha}2^{-\beta}$$

$$= (\frac{n+1}{(n+2)(2n+1)})^{\alpha}2^{-\beta}(n+1)^{-\beta}.$$

Which contradicts (9). Hence $|w(z)| < 1$ and $f \in C_n$.

References:

[1] GANIGI M D and URALEGADDI B A. *New criteria for meromorphic univalent functions* [J]. Bull. Math. de la Soc. Sci. Math. de la R.S. de Roumanie, 33(81), 1989, **1**: 9–13.

[2] JACK I S. *Functions starlike and convex of order* α[J]. J. London Math. Soc., 1971, **2**(3): 469–474.

[3] OBRADOVIC M. *On some criteria for starlikeness in the unit disc* [J], FILOMAT (Nis), 1993, **7**: 87–91.

[4] SARANGI S M and SUGUNA B U. *Certain sufficient conditions for close-to-convexity and starlikeness of meromorphic functions* [J]. To appear in Maths Student.

[5] SARANGI S M and SUGUNA B U. *New criteria for meromorphic close-to-convex functions* [M]. Submitted for publication.