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An Inequality for the Determinant of the GCD Matrix and the GCUD Matrix *

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Abstract: Let $S = \{x_1, x_2, \ldots, x_n\}$ be a set of distinct positive integers. The $n \times n$ matrix (S) whose i, j-entry is the greatest common divisor (x_i, x_j) of x_i and x_j is called the GCD matrix on S. A divisor d of x is said to be a unitary divisor of x if $(d, x/d) = 1$. The greatest common unitary divisor (GCUD) matrix (S^{**}) is defined analogously. We show that if S is both GCD-closed and GCUD-closed, then $\det(S^{**}) \geq \det(S)$, where the equality holds if and only if $(S^{**}) = (S)$.

Key words: Smith's determinant; GCD matrix; unitary analogue; inequality.

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1. Introduction

Let $S = \{x_1, x_2, \ldots, x_n\}$ be a set of distinct positive integers. The $n \times n$ matrix (S) whose i, j-entry is the greatest common divisor (x_i, x_j) of x_i and x_j is called the GCD matrix on S (see [1]).

A divisor d of x is said to be a unitary divisor of x if $(d, x/d) = 1$. If d is a unitary divisor of x, we write $d \| x$. The greatest common unitary divisor of x_i and x_j is denoted by $(x_i, x_j)^{**}$. The $n \times n$ matrix (S^{**}) whose i, j-entry is the greatest common unitary divisor $(x_i, x_j)^{**}$ is called the GCUD matrix on S (see [3]).

The set S is said to be factor-closed if it contains every divisor of any element of S, and the set S is said to be GCD-closed if it contains the greatest common divisor of any two elements of S. Unitary divisor -closed (UD-closed) sets and GCUD-closed sets are defined analogously. It can be verified that

i) S is factor-closed $\Rightarrow$ S is GCD-closed,

ii) S is factor-closed $\Rightarrow$ S is UD-closed $\Rightarrow$ S is GCUD-closed.

H.J.S.Smith[8] showed that if S is factor-closed, then

$$\det(S) = \prod_{k=1}^{n} \phi(x_k), \tag{1}$$

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where ϕ is Euler's totient function. Beslin and Ligh[2] showed that if S is GCD-closed, then

$$\det(S) = \prod_{k=1}^{n} \sum_{\substack{d \mid x_k \\ d \nmid x_t \\ x_t < x_k}} \phi(d). \tag{2}$$

If S is factor-closed, then (2) reduces to (1).

It is known[3] that if S is UD-closed, then

$$\det(S^{**}) = \prod_{k=1}^{n} \phi^*(x_k), \tag{3}$$

where ϕ^* is the unitary analogue of Euler's totient function [7]. It is also known [3] that if S is GCUD-closed, then

$$\det(S^{**}) = \prod_{k=1}^{n} \sum_{\substack{d \| x_k \\ d \nparallel x_t \\ x_t < x_k}} \phi^*(d). \tag{4}$$

If S is UD-closed, then (4) reduces to (3).

Euler's totient function ϕ and its unitary analogue ϕ^* are multiplicative functions such that for all prime powers p^a (> 1)

$$\begin{aligned} \phi(p^a) &= p^a - p^{a-1}, \\ \phi^*(p^a) &= p^a - 1. \end{aligned}$$

Thus $\phi^*(x) \geq \phi(x)$ for all positive integers x. If S is a factor-closed set, then (1) and (3) hold and further

$$\det(S^{**}) \geq \det(S). \tag{5}$$

In this paper we show that if S is both GCD-closed and GCUD-closed, then (5) holds, and the equality in (5) holds if and only if $(S^{**}) = (S)$, see Section 2.

If S is factor-closed, then S is both GCD-closed and GCUD-closed. Therefore (5) for factor-closed sets is a special case of our result. Note that there also exists an infinite number of sets S which are both GCD-closed and GCUD-closed but which are not factor-closed. For example, all sets of the form $S_1 = \{p, pq\}$ and $S_2 = \{1, p, p^3\}$, where p and q are distinct primes, are both GCD-closed and GCUD-closed but are not factor-closed. Here $(S_1) = (S_1^{**})$ and $(S_2) \neq (S_2^{**})$.

At the end of this paper we discuss briefly another unitary analogue of the GCD matrix than the GCUD matrix, see Section 3.

2. The main results

Theorem 2.1 *If S is both GCD-closed and GCUD-closed, then*

$$\det(S^{**}) \geq \det(S).$$

Proof We use the evaluations (2) and (4). Let k be fixed. Let

$$\begin{aligned} P_k &= \{d : d \mid x_k, d \mid x_t \text{ for some } x_t < x_k\}, \\ P_k^* &= \{d : d \| x_k, d \| x_t \text{ for some } x_t < x_k\}. \end{aligned}$$

Let $M_k^* = \{d_1, d_2, \ldots, d_m\}$ denote the set of maximal elements of P_k^* under the partial ordering induced by the unitary divisibility. It is evident that

$$\begin{aligned} M_k^* &\subseteq P_k^* \subseteq P_k, \\ d \in P_k &\Rightarrow D(d) \subseteq P_k, \\ d \in P_k^* &\Rightarrow U(d) \subseteq P_k^*, \end{aligned}$$

where $D(d)$ is the set of divisors of d and $U(d)$ is the set of unitary divisors of d. Since $d_1, d_2, \ldots, d_m \in P_k$, it can be seen that

$$P_k \supseteq \bigcup_{i=1}^{m} D(d_i). \tag{6}$$

By definition of the set $M_k^* = \{d_1, d_2, \ldots, d_m\}$, it can be seen that

$$P_k^* = \bigcup_{i=1}^{m} U(d_i). \tag{7}$$

It is well known that

$$\sum_{d \mid x_k} \phi(d) = \sum_{d \| x_k} \phi^*(d) = x_k. \tag{8}$$

Therefore,

$$\begin{aligned} \sum_{\substack{d \mid x_k \\ d \nmid x_t \\ x_t < x_k}} \phi(d) &= x_k - \sum_{d \in P_k} \phi(d), \\ \sum_{\substack{d \| x_k \\ d \not\| x_t \\ x_t < x_k}} \phi^*(d) &= x_k - \sum_{d \in P_k^*} \phi^*(d). \end{aligned}$$

Thus, by (2) and (4), it is enough to prove that

$$\sum_{d \in P_k} \phi(d) \geq \sum_{d \in P_k^*} \phi^*(d). \tag{9}$$

Consider the sum $\sum_{d \in P_k^*} \phi^*(d)$. By (7) and a method similar to the inclusion-exclusion principle,

$$\begin{aligned} \sum_{d \in P_k^*} \phi^*(d) &= \sum_{d \in \bigcup_{i=1}^{m} U(d_i)} \phi^*(d) \\ &= \sum_{1 \leq i \leq m} \sum_{d \| d_i} \phi^*(d) - \sum_{1 \leq i < j \leq m} \sum_{d \| (d_i, d_j)^{**}} \phi^*(d) \\ &\quad + \cdots + (-1)^{m+1} \sum_{d \| (d_1, d_2, \ldots, d_m)^{**}} \phi^*(d). \end{aligned} \tag{10}$$

Since the d_i's are unitary divisors of x_k, the GCUD's in (10) are equal to the GCD's. Therefore application of (8) gives

$$\sum_{d\in P_k^*} \phi^*(d) = \sum_{1\le i\le m} d_i - \sum_{1\le i<j\le m} (d_i, d_j) + \cdots + (-1)^{m+1}(d_1, d_2, \ldots, d_m). \tag{11}$$

Next, consider the sum $\sum_{d\in P_k} \phi(d)$. By (6) and a method similar to the inclusion-exclusion principle,

$$\begin{aligned}\sum_{d\in P_k} \phi(d) &\ge \sum_{d\in\bigcup_{i=1}^m D(d_i)} \phi(d)\\ &= \sum_{1\le i\le m} \sum_{d|d_i} \phi(d) - \sum_{1\le i<j\le m} \sum_{d|(d_i,d_j)} \phi(d)\\ &\quad + \cdots + (-1)^{m+1} \sum_{d|(d_1,d_2,\ldots,d_m)} \phi(d).\end{aligned} \tag{12}$$

Application of (8) shows that the right-hand side of (12) is equal to the right-hand side of (11). This shows that (9) holds. □

Theorem *If S is both GCD-closed and GCUD-closed, then*

$$\det(S) = \det(S^{**}) \Leftrightarrow (S) = (S^{**}).$$

Proof If $(S) = (S^{**})$, then $\det(S) = \det(S^{**})$. We assume that $(S) \ne (S^{**})$ and prove that $\det(S^{**}) > \det(S)$. By (9), it is enough to prove that

$$\sum_{d\in P_k} \phi(d) > \sum_{d\in P_k^*} \phi^*(d)$$

or

$$P_k \supsetneq \bigcup_{i=1}^m D(d_i) \tag{13}$$

for some $k = 1, 2, \ldots, n$. Suppose that x_k is the smallest number such that

$$\exists x_t < x_k : (x_t, x_k)^{**} \ne (x_t, x_k). \tag{14}$$

If there are many such x_t's, we take the smallest of them. We show that $(x_t, x_k) \in P_k$ but $(x_t, x_k) \notin \bigcup_{i=1}^m D(d_i)$. It is clear that $(x_t, x_k) \in P_k$. Further, suppose on the contrary that $(x_t, x_k) \in \bigcup_{i=1}^m D(d_i)$. Then there exists i such that

$$(x_t, x_k) \mid d_i. \tag{15}$$

By the definition of d_i, we have

$$d_i \| x_k,\ d_i \| x_r \text{ for some } x_r < x_k. \tag{16}$$

We show that $(x_t, x_k) = x_t$. If $(x_t, x_k) < x_t$, then, by minimality of x_k and x_t in (14),

$$(x_t, x_k) = ((x_t, x_k), x_t) = ((x_t, x_k), x_t)^{**},$$
$$(x_t, x_k) = ((x_t, x_k), x_k) = ((x_t, x_k), x_k)^{**}.$$

This implies that the common prime divisors of x_k and x_t must occur in the same power, which contradicts (14). Therefore $(x_t, x_k) = x_t$.

Now (15) and (16) can be written as

$$x_t \mid d_i \| x_r, \ d_i \| x_k, \ x_r < x_k.$$

Since $x_t \not\| x_k$ (see (14)), we have $x_t \not\| d_i$. Therefore

$$(x_t, x_r)^{**} = (x_t, d_i)^{**} < x_t = (x_t, x_r).$$

Thus $(x_t, x_r)^{**} \neq (x_t, x_r)$. This contradicts the minimality of x_k in (14). So we have proved that $(x_t, x_k) \notin \bigcup_{i=1}^{m} D(d_i)$ and further that (13) holds. □

Example 2.1 Let $S_1 = \{2, 6\}$ and $S_2 = \{1, 2, 8\}$. Then S_1 and S_2 are GCD-closed and GCUD-closed but not factor-closed. Further, $(S_1) = (S_1^{**})$ and $(S_2) \neq (S_2^{**})$. Note that $\det(S_1) = \det(S_1^{**}) = 8$ but $\det(S_2) = 6 < \det(S_2^{**}) = 7$.

Example 2.2 Let $S_1 = \{1, 2\}$ and $S_2 = \{1, 2, 4\}$. Then S_1 and S_2 are factor-closed and therefore also GCD-closed and GCUD-closed. Further, $(S_1) = (S_1^{**})$ and $(S_2) \neq (S_2^{**})$. Note that $\det(S_1) = \det(S_1^{**}) = 1$ but $\det(S_2) = 2 < \det(S_2^{**}) = 3$.

3. Another unitary analogue

The semi-unitary greatest common divisor (SUGCD) of x_i and x_j is defined as the greatest unitary divisor of x_j which is a divisor of x_i. The SUGCD of x_i and x_j is denoted by $(x_i, x_j)^*$. The $n \times n$ matrix (S^*) whose i, j-entry is equal to $(x_i, x_j)^*$ is said to be the SUGCD matrix on S. It can be verified that

$$S \text{ is UD-closed} \Rightarrow S \text{ is SUGCD-closed} \Rightarrow S \text{ is GCUD-closed}.$$

If S is GCD-closed, then

$$S \text{ is SUGCD-closed} \Leftrightarrow S \text{ is GCUD-closed}.$$

It is known [3, Remark 5.5] that if S is SUGCD-closed, then

$$\det(S^*) = \det(S^{**}).$$

Therefore, by Theorem 2.2, if S is both SUGCD-closed and GCD-closed, then

$$\det(S^*) \geq \det(S), \tag{17}$$

and the equality holds if and only if

$$(S^{**}) = (S). \tag{18}$$

However, we feel that the last result is not satisfactory. One should find a characterization for the equality in (17) other than (18).

Remark The methods of this paper apply to regular arithmetical convolutions[6] so that the unitary convolution U and the Dirichlet convolution D, respectively, could be replaced with regular arithmetical convolutions A and B with $A \leq B$, where $\leq$ is the partial ordering by McCarthy ([4], [5,p.169]). For the sake of brevity we do not present the details here.

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