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A New Demonstration of a Theorem for Sublinear Approximations *

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Abstract: The demonstration of a theorem for sublinear approximations, due to Demyanov and Rubinov, is modified as a new one in this paper.

Key words: sublinear; superlinear; approximation.

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In the study of extreme problems, it is sometimes more convenient to consider an approximation that is not an upper (lower) convex approximation but a majorant (minority) of the function itself. Recently, Demyanov and Rubinov gave an exact majorant of a positively homogeneous function at a given point which is stated as follows.

Theorem ([1], pp 169) *Let a function $h : \mathbb{R}^n \to \mathbb{R}$ be positively homogeneous of first degree, $z \in \mathbb{R}^n$ and $z \neq 0$. Then the following conditions are equivalent:*

(a) There exists a sublinear function $p(\cdot)$ such that $p(x) \geq h(x)$ $(\forall x)$ and $p(z) = h(z)$.

(b) There exists a constant $k > 0$ such that $h(x) - h(z) \leq k\|x - z\|(\forall x)$ holds.

In the proof of the part $(b) \Rightarrow (a)$, $p(x) = \frac{\|x\|}{\|z\|}h(z) + \frac{k}{\|z\|}\|\|z\|x - \|x\|z\|$. In fact, the above function is not sublinear. For example, set $e_1 = (1,0,0,\cdots,0)^T, e_2 = (0,1,0,\cdots,0)^T \in \mathbb{R}^n$ and consider the case in which $h(\cdot) = -\|\cdot\|$ and $z = e_1$. Then h is positively homogeneous and the k in (b) can be taken as 1. It is easily to prove that $p(e_1 + e_2) > p(e_1) + p(e_2)$, i.e., $p(\cdot)$ is not a sublinear function. We now give a modified demontration of the theorem.

Proof of $(b) \Rightarrow (a)$: Assume that (b) holds.

(1) If $h(z) \geq 0$, then the function $g(x) = h(z) + k\|x - z\|$ is convex and for any $x \in \mathbb{R}^n$, one has $g(x) \geq 0, g(x) \geq h(x), g(z) = h(z)$.

Definite $p(x) = \inf_{s>0} \sup_{t \in \mathrm{co}(s,1/s)} \frac{\|x\|}{\|z\|} g\left(\frac{t\|z\|}{\|x\|}x\right)$ for $x \neq 0$ and $p(0) = 0$. Then p is positively homogeneous and $p(z) = h(z)$,

$$p(x) \geq \frac{\|x\|}{\|z\|} g\left(\frac{\|z\|}{\|x\|}x\right) \geq \frac{\|x\|}{\|z\|} h\left(\frac{\|z\|}{\|x\|}x\right) = h(x).$$

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In addition, p satifies subadditivity. In fact, if $x+y=0$, then we have

$$p(x)+p(y) \geq \frac{\|x\|}{\|z\|} g\left(\frac{\|z\|}{\|x\|} x\right)+\frac{\|y\|}{\|z\|} g\left(\frac{\|z\|}{\|y\|} y\right) \geq 0=p(x+y)$$

If $x+y \neq 0$, then

$$\begin{aligned}
&\tfrac{\|x+y\|}{\|z\|} g\left(\tfrac{t\|z\|}{\|x+y\|}(x+y)\right) \\
&\leq \tfrac{\|x\|+\|y\|}{\|z\|} g\left(\tfrac{\|x\|}{\|x\|+\|y\|} \tfrac{\|x\|+\|y\|}{\|x+y\|} \tfrac{t\|z\|}{\|x\|} x+\tfrac{\|y\|}{\|x\|+\|y\|} \tfrac{\|x\|+\|y\|}{\|x+y\|} \tfrac{t\|z\|}{\|y\|} y\right) \\
&\leq \tfrac{\|x\|+\|y\|}{\|z\|}\left(\tfrac{\|x\|}{\|x\|+\|y\|} g\left(\tfrac{\|x\|+\|y\|}{\|x+y\|} \tfrac{t\|z\|}{\|x\|} x\right)+\tfrac{\|y\|}{\|x\|+\|y\|} g\left(\tfrac{\|x\|+\|y\|}{\|x+y\|} \tfrac{t\|z\|}{\|y\|} y\right)\right) \\
&\leq \sup_{r \in \operatorname{co}(s_1, 1/s_1)} \tfrac{\|x\|}{\|z\|} g\left(\tfrac{r\|z\|}{\|x\|} x\right)+\sup_{r \in \operatorname{co}(s_1, 1/s_1)} \tfrac{\|y\|}{\|z\|} g\left(\tfrac{r\|z\|}{\|y\|} y\right):=I_{s_1},
\end{aligned}$$

where $s>0, t \in \operatorname{co}(s, 1/s), s_1=\frac{\|x\|+\|y\|}{\|x+y\|} s$. Since t is arbitrary, we have $p(x+y) \leq \sup_{t \in \operatorname{co}(s, 1/s)} \frac{\|x+y\|}{\|z\|} g\left(\frac{t\|z\|}{\|x+y\|}(x+y)\right) \leq I_{s_1}$. Therefore, one obtains $p(x+y) \leq \inf_{s>0} I_{s_1}=\inf_{s_1>0} I_{s_1}=p(x)+p(y)$. It is shown that p satisfies (a).

(2) Suppose that $h(z)<0$. Since $z=(z_1, z_2, \cdots, z_n)^T \neq 0$, we take some $z_i \neq 0$ and definite $h_1(x)=h(x)-\frac{x_i h(z)}{z_i}, \forall x=(x_1, x_2, \cdots, x_n)^T \in \mathbb{R}^n$. Then h_1 is positively homogeneous, $h_1(z)=0$ and for any $x \in \mathbb{R}^n$, $h_1(x)-h_1(z) \leq (k+|h(z)/z_i|)\|x-z\|$. By the conclusion of (1), there exists a sublinear function p_1 such that $h_1(x) \leq p_1(x)$ and $p_1(z)=h_1(z)$. Then the function $p(x)=p_1(x)+x_i h(z)/z_i$ is also sublinear and satisfies $h(z)=p(z), h(x) \leq p(x), \forall x \in \mathbb{R}^n$. This implies that (a) holds. □

Corollary *Assume that $h: \mathbb{R}^n \rightarrow \mathbb{R}$ is Lipschtzian and positively homogeneous, $z \in \mathbb{R}^n$ and $z \neq 0$. Then there exist a sublinear function p and a superlinear function q such that $q(x) \leq h(x) \leq p(x) (\forall x \in \mathbb{R}^n)$ and $q(z)=p(z)=h(z)$.*

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关于一个次线性逼近定理的新证法

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摘 要: Demyanov 和 Rubinov 在 [1] 中给出了一个次线性逼近定理. 本文对该定理的证明进行了修正.