Journal of Mathematical Research & Exposition
Vol.20, No.3, 331-336, August 2000

# Convergence Theorems of Iterative Sequences for Asymptotically Non-Expansive Mapping in a Uniformly Convex Banach Space *

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**Abstract:** In this paper, Jürgen's relative result is extended to a uniformly convex Banach space, and the convergence of iterative sequence in an uniformly convex Banach space for asymptotically non-expansive mapping is proved.

**Key words:** uniformly convex; convergence; mapping; iterate.

**Classification:** AMS(1991) 41A36/CLC O174.41

**Document code:** A **Article ID:** 1000-341X(2000)03-0331-06

Let $C$ be a nonempty subset of a linear normal space.

1. A self-mapping $T : C \to C$ is called asymptotically non-expansive with sequence $\{K_n\} \iff \lim\limits_{n\to+\infty} K_n = 1, K_n \geq 1$, and $\|T^n x - T^n y\| \leq K_n\|x-y\|, \forall n \in N, \forall x, y \in C$;

2. A self-mapping $T : C \to C$ is called uniformly Lipschitzian $\iff \|T^n x - T^n y\| \leq L\|x-y\|$, for some constant $L > 0$ and $\forall x, y \in C, \forall n \in N$.

In [1], [2], we proved convergence theorems of iterative sequences in a Hilbert space. Jürgen Schu[3] studied the convergence of Mann iterative sequences for asymptotically non-expansive mappings in a Hilbert space. In this paper, we will prove the convergence of iterative sequence in an uniformly convex Banach space for asymptotically non-expansive mappings and extend Jürgen's relative results to a uniformly convex Banach space. The following theorems will be proved.

**Theorem 1** *Let $T$ be a completely continuously asymptotically non-expansive mapping with sequence $\{K_n\}$ in a bounded closed convex subset $C$ of a uniformly convex Banach space and $K_n \geq 1$, $\sum\limits_{n=1}^{+\infty}(K_n - 1) < +\infty, x_0 \in C$. If*

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T^n y_n, y_n = \beta_n T^n x_n + (1-\beta_n)x_n,$$

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where $\{\alpha_n\}$ and $\{\beta_n\}$ satisfy $0 < \alpha \le \alpha_n \le \frac{1}{2}$, $0 \le \beta_n \le \frac{1}{2}$, and $\lim\limits_{n\to+\infty} \beta_n = 0$, then the iterative sequence $(x_n)$ converges to a fixed point $p$ of $T$.

**Theorem 2** *Let $T$ be a completely continuously asymptotically non-expansive mapping with sequence $\{K_n\}$ in a bounded closed convex subset $C$ of uniformly convex Banach space and $K_n \ge 1$, $\sum\limits_{n=1}^{+\infty}(K_n - 1) < +\infty, x_0 \in C$. If*

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T^n x_n,$$

*where $\{\alpha_n\}$ satisfies that $0 < \alpha \le \alpha_n \le \frac{1}{2}$, then the iterative sequence $\{x_n\}$ converges to a fixed point $p$ of $T$.*

In order to prove the above theorems, the following lemma given by Jűrgen Schu[3] and a new lemma to be proved will be useful.

**Jűrgen's Lemma** *Let $C$ be a nonempty convex subset of a linear normal space and $T : C \to C$ is a uniformly Lipschitzian mapping, if*

$$\begin{aligned} x_{n+1} &= (1 - \alpha_n)x_n + \alpha_n T^n y_n, \quad y_n = \beta_n T^n x_n + (1 - \beta_n)x_n, \\ C_n &= \|T^n x_n - x_n\|, \end{aligned}$$

*then $\|Tx_n - x_n\| \le C_n + C_{n-1}L(1 + 3L + 2L^2), \forall n \in N$.*

**Lemma** *Let $C$ be a nonempty bounded closed convex subset of a uniformly convex Banach space, $T : C \to C$ an asymptotically non-expansive mapping with sequence $(K_n)$, $F(T)$ the set of fixed point of $T$, $K_n \ge 1$, $\sum\limits_{n=1}^{+\infty}(K_n - 1) < +\infty$, and $F(T)$ nonempty. If $x_0 \in C$ and*

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T^n y_n, \ y_n = \beta_n T^n x_n + (1 - \beta_n)x_n,$$

*where $(\alpha_n)$ and $\beta_n)$ satisfy $0 < \alpha \le \alpha_n \le \frac{1}{2}$, $0 \le \beta_n \le \frac{1}{2}$, and $\lim\limits_{n\to+\infty} \beta_n = 0$, then*

$$\lim_{n\to+\infty} \|Tx_n - x_n\| = 0.$$

**Proof of Lemma** First, we prove $\lim\limits_{n\to+\infty} \|x_n - T^n y_n\| = 0$. If not, there must exist a subsequence $\{n_k\}_{k=1}^{+\infty}$ of $\{n\}_{n=1}^{+\infty}$ and $\overline{\varepsilon_0} > 0$, such that

$$\|x_{n_k} - T^{n_k} y_{n_k}\| \ge \overline{\varepsilon_0}. \tag{1}$$

It is obvious that for $\forall p \in F(T)$

$$\|x_{n_k} - T^{n_k} y_{n_k}\| \le \|x_{n_k} - p\| + \|T^{n_k} y_{n_k} - p\| \le \|x_{n_k} - p\| + K_{n_k}\|y_{n_k} - p\|, \tag{2}$$

$$\begin{aligned} \|y_n - p\| &= \|(1 - \beta_n)x_n + \beta_n T^n x_n - p\| \\ &\le (1 - \beta_n)\|x_n - p\| + \beta_n\|T^n x_n - p\| \\ &\le (1 - \beta_n)\|x_n - p\| + K_n\beta_n\|x_n - p\| = [1 + (K_n - 1)\beta_n]\|x_n - p\|. \end{aligned} \tag{3}$$

From (2) and (3), it follows that

$$\begin{aligned}\|x_{n_k} - T^{n_k}y_{n_k}\| &\le \|x_{n_k} - p\| + K_{n_k}[1 + (K_{n_k} - 1)\beta_{n_k}]\|x_{n_k} - p\| \\ &= [1 + K_{n_k} + K_{n_k}(K_{n_k} - 1)\beta_{n_k}]\|x_{n_k} - p\|. \end{aligned} \tag{4}$$

Since $\{K_n\}$ is convergent, there is some $M_1 > 0$ such that

$$|1 + K_{n_k} + K_{n_k}(K_{n_k} - 1)\beta_{n_k}| \le M_1.$$

From (4),

$$\|x_{n_k} - T^{n_k}y_{n_k}\| \le M_1\|x_{n_k} - p\|, \quad \forall p \in F(T).$$

From (1), we have

$$\|x_{n_k} - p\| \ge \frac{\overline{\varepsilon_0}}{M_1} = \varepsilon_0 > 0, \quad \forall p \in F(T). \tag{5}$$

Besides, $\|T^n y_n - p\| \le K_n\|y_n - p\|$, thus from (3)

$$\|T^n y_n - p\| \le K_n[1 + (K_n - 1)\beta_n]\|x_n - p\|.$$

Setting $K_n[1 + (K_n - 1)\beta_n] = a_n$, we obtain

$$\|T^n y_n - p\| \le a_n\|x_n - p\|, \qquad \forall p \in F(T). \tag{6}$$

Because of the convergence of $\{K_n\}$, $0 \le \beta_n \le \frac{1}{2}$ and the boundedness of $C$, there nust exist some constant $M_2 > 0$ such that

$$a_n\|x_n - p\| = [K_n + K_n(K_n - 1)\beta_n]\|x_n - p\| \le M_2. \tag{7}$$

It follows from (1) and (7) that

$$\left\|\frac{x_{n_k} - p}{\alpha_{n_k}\|x_{n_k} - p\|} - \frac{T^{n_k}y_{n_k} - p}{\alpha_{n_k}\|x_{n_k} - p\|}\right\| = \frac{\|x_{n_k} - T^{n_k}y_{n_k}\|}{\alpha_{n_k}\|x_{n_k} - p\|} \ge \frac{\overline{\varepsilon_0}}{M_2} > 0, \tag{8}$$

$\beta_n \ge 0$ and $K_n \ge 1$ give that $a_n = K_n + K_n(K_n - 1)\beta_n \ge 1$. So we get $\left\|\frac{x_{n_k} - p}{\alpha_{n_k}\|x_{n_k} - p\|}\right\| \le 1$.

However, from (6), we have $\left\|\frac{T^{n_k}y_{n_k} - p}{\alpha_{n_k}\|x_{n_k} - p\|}\right\| \le 1$ . Thus, by the uniform convexity of the space, there must exist $\delta = \delta(\frac{\overline{\varepsilon_0}}{M_2}) > 0$ such that

$$\begin{aligned}\|x_{n_{k+1}} - p\| &= \|(1 - \alpha_{n_k})x_{n_k} + \alpha_{n_k}T^{n_k}y_{n_k} - p\| \\ &\le (1 - 2\alpha_{n_k})\|x_{n_k} - p\| + \|\alpha_{n_k}(x_{n_k} - p) + \alpha_{n_k}(T^{n_k}y_{n_k} - p)\| \\ &\le (1 - 2\alpha_{n_k})\|x_{n_k} - p\| + \alpha_{n_k}a_{n_k}\|x_{n_k} - p\|\left\|\frac{x_{n_k} - p}{a_{n_k}\|x_{n_k} - p\|} + \frac{T^{n_k}y_{n_k} - p}{a_{n_k}\|x_{n_k} - p\|}\right\| \\ &\le (1 - 2\alpha_{n_k})\|x_{n_k} - p\| + \alpha_{n_k}a_{n_k}\|x_{n_k} - p\|(2 - \delta) \\ &= [1 + 2\alpha_{n_k}(a_{n_k} - 1) - \delta\alpha_{n_k}a_{n_k}]\,\|x_{n_k} - p\|, \quad \forall p \in F(T). \end{aligned} \tag{9}$$

Since

$$a_n - 1 = K_n + K_n(K_n - 1)\beta_n - 1 = (K_n - 1)(1 + K_n\beta_n), \tag{10}$$

from the boundedness of $C$ and $0 < \alpha_n \le 1$ , there must exist some $M_3 > 0$ such that

$$2\alpha_n(1 + K_n\beta_n)||x_n - p|| \le M_3. \tag{11}$$

Substituting (10) and (11) into (9), we obtian that

$$||x_{n_{k+1}} - p|| \le ||x_{n_k} - p|| + M_3(K_{n_k} - 1) - \delta\alpha_{n_k}a_{n_k}||x_{n_k} - p||. \tag{12}$$

It follows from (5) that $||x_{n_k} - p|| \ge \varepsilon_0 > 0$. However, $a_n = K_n + K_n(K_n - 1)\beta_n \ge 1$ and $0 < \alpha \le \alpha_n$. Hence, from (12)

$$||x_{n_{k+1}} - p|| \le ||x_{n_k} - p|| + M_3(K_{n_k} - 1) - \delta\alpha\varepsilon_0, \tag{13}$$

$$||x_{n+1} - p|| = ||(1 - \alpha_n)x_n + \alpha_n T^n y_n - p|| \le (1 - \alpha_n)||x_n - p|| + \alpha_n||T^n y_n - p||$$

From (6),

$$||x_{n+1} - p|| \le (1 - \alpha_n)||x_n - p|| + \alpha_n a_n||x_n - p|| = [1 - (a_n - 1)\alpha_n]||x_n - p||. \tag{14}$$

From (10),

$$||x_{n+1} - p|| \le [1 + (1 + K_n\beta_n)(K_n - 1)\alpha_n]||x_n - p||. \tag{15}$$

Because $C$ is bounded and $0 < \alpha_n \le 1$, there exists some constant $M > M_3 > 0$, such that

$$(1 + K_n\beta_n)\alpha_n||x_n - p|| \le M.$$

Thus from (15), it can be obtained that

$$||x_{n+1} - p|| \le ||x_n - p|| + M(K_n - 1). \tag{16}$$

By (16), (13) will become

$$\begin{aligned}
||x_{n_k+1} - p|| &\le ||x_{n_k} - p|| + M(K_{n_k} - 1) - \delta\alpha\varepsilon_0 \\
&\le ||x_{n_k-1} - p|| + M(K_{n_k} - 1) - \delta\alpha\varepsilon_0 \le \cdots \\
&\le ||x_{n_{k-1}+1} - p|| + M\sum_{\rho=n_{k-1}+1}^{n_k}(K_\rho - 1) - \delta\alpha\varepsilon_0 \\
&\le ||x_{n_{k-1}} - p|| + M\sum_{\rho=n_{k-1}}^{n_k}(K_\rho - 1) - \delta\alpha\varepsilon_0 \times 2 \le \cdots \\
&\le ||x_{n_1} - p|| + M\sum_{\rho=n_1}^{n_k}(K_\rho - 1) - k\delta\alpha\varepsilon_0 \\
&\le ||x_{n_1} - p|| + M\sum_{\rho=1}^{+\infty}(K_\rho - 1) - k\delta\alpha\varepsilon_0.
\end{aligned} \tag{17}$$

Note $\sum\limits_{\rho=1}^{+\infty}(K_\rho - 1) < +\infty$ and $\delta\alpha\varepsilon_0 > 0$. Thus, for sufficiently large $k$, we have from (17) that $||x_{n_k+1} - p|| < 0$. It is a contradiction. Therefore, $\lim\limits_{n\to+\infty}||x_n - T^n y_n|| = 0$.

On the other hand

$$\begin{aligned}\|x_n - T^n x_n\| &\le \|x_n - T^n y_n\| + \|T^n y_n - T^n x_n\| \\ &\le \|x_n - T^n y_n\| + K_n\|y_n - x_n\| \\ &= \|x_n - T^n y_n\| + K_n\beta_n\|x_n - T^n x_n\|.\end{aligned}$$

Noting that $\{K_n\}$ is convergent and $\lim\limits_{n\to+\infty}\beta_n = 0$, there is some constant $N$ such that $K_n\beta_n \le \frac{1}{2}$ when $n > N$. Thus

$$\|x_n - T^n x_n\| \le \|x_n - T^n y_n\| + \frac{1}{2}\|x_n - T^n x_n\|,$$

that is $\|x_n - T^n x_n\| \le 2\|x_n - T^n y_n\|$ when $n > N$.

So $\lim\limits_{n\to+\infty}\|x_n - T^n y^n\| = 0$. implies $\lim\limits_{n\to+\infty}\|x_n - T^n x_n\| = 0$. On the other hand, since $T$ is asymptotically non-expansive, $T$ must be uniformly Lipschitzian. Therefore, using Jűrgen's Lemma, $\lim\limits_{n\to+\infty}\|x_n - Tx^n\| = 0$. This completes the proof of the Lemma.

**Proof of Theorem 1** From the Schauder's theorem, it follows that $F(T)$ is non-empty. Hence, by Lemma,

$$\lim_{n\to+\infty}\|x_n - Tx_n\| = 0. \tag{18}$$

Since $T$ is completely continuous and $C$ is a closed bounded subset, $\{Tx_n\}_{n=1}^{+\infty}$ must have a convergent subset $\{Tx_{n_k}\}_{k=1}^{+\infty}$ . Set

$$\lim_{k\to+\infty} Tx_{n_k} = p. \tag{19}$$

It follows from (18) and (19) that

$$\lim_{k\to+\infty} x_{n_k} = p, \tag{20}$$

$T$ is completely continuous implies that $T$ must be continuous. Noting (18) and (20), we can obtain $\|p - Tp\| = 0$ , that is, $p$ is a fixed point, It follows from (16) that

$$\|x_{n+m} - p\| \le \|x_n - p\| + \sum_{\rho=n}^{n+m-1} M(K_\rho - 1). \tag{21}$$

Now, we prove $\lim\limits_{n\to+\infty} x_n = p$. For any given $\varepsilon > 0$, since $\sum\limits_{\rho=1}^{+\infty}(K_\rho - 1) < +\infty$ there exists some natural number $N$ such that

$$M\sum_{\rho=n}^{+\infty}(K_\rho - 1) < \frac{\varepsilon}{2}, \qquad \forall n \ge N. \tag{22}$$

But from (20), there is some natural number $k_0 \ge N$ such that

$$\|x_{n_k} - p\| < \frac{\varepsilon}{2}, \qquad \forall k \ge k_0. \tag{23}$$

It follows from (21), (22) and (23) that

$$\|x_{n_{k_0}+m} - p\| \le \|x_{n_{k_0}} - p\| + \sum_{\rho=n_{k_0}}^{+\infty} M(K_\rho - 1) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \qquad \forall m \in N.$$

Therefore, $\lim_{n\to+\infty} \|x_n - p\| = 0$, that is, $\lim_{n\to+\infty} x_n = p$. This completes the proof of Theorem 1.

Theorem 2 can be proved by taking $\beta_n = 0$ in the Theorems 1.

**Acknowledgement** We express our sincere thanks for the financial support from the Scientific Fund of Aeronautics and Astronautics Ministry of P.R.China.

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# 一致凸的 Banach 空间上的渐近非扩张映象的迭代序列的收敛性定理

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**摘 要**: 本文把 [3] 的主要结果从 Hilbert 空间推广到一致凸的 Banach 空间，证明了一致凸的 Banach 空间上的渐近非扩张映象的迭代序列的收敛性.