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On the Product of Two Nilpotent Subgroups of a Finite Group *

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Abstract: It is known that the product of two nilpotent subgroups of a finite group is not necessarily nilpotent. In this paper , we study the influence of the Engel condition on the product of two nilpotent subgroups. Our results generalize some well-known results.

Key words: n-Engel condition; nilpotent group; p-nilpotent group.

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1. Basic results and Notation

We shall use the following notation for commutators:

$$[x, y] = x^{-1}y^{-1}xy$$
$$[x_1, x_2, \cdots, x_n] = [[x_1, \cdots, x_{n-1}], x_n] \quad (n \geq 3),$$
$$[x, {}_0y] = x,$$
$$[x, {}_ny] = [[x, {}_{n-1}y],] \quad (n \geq 1)$$

$[x, {}_ny] = 1$ is called n-Engel condition. The rest of the notation is standard(see[4]). In this paper, all groups considered are finite.

We shall need the following results:

Lemma 1 *Assume that $G = AB$, that A and B are nilpotent subgroups of G, $[A, B] = 1$, then G is nilpotent.*

Proof By [2, Theorem 2. 5, P. 122], it is obvious.

Lemma 2 *Assume that $G = AB$, A is a normal nilpotent subgroup of G, B is a nilpotent subgroup of G, $(|A|, |B|) = 1$, if for each element x in A and each element y in B there is*

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a positive integer n such that $[x,_n y] = 1$, then G is nilpotent.

Proof We may assume that $n \geq 2$. Let $a = [x,_{n-2} y]$. Then $[a, y, y] = 1$. Since $[(y^{-1})^a y, y] = [a, y, y] = 1$, we have $y(y^{-1})^a y = (y^{-1})^a yy$, $y(y^{-1})^a = (y^{-1})^a y$, that is $[(y^{-1})^a, y] = 1$. Thus $o((y^{-1})^a y) \mid o(y)$, $o(y) \mid |B|$. Since $A \lhd G$, so that $(y^{-1})^a y = [a, y] \in A$. By hypothesis, $(| A |, | B |) = 1$, we have $[a, y] = [x,_{n-1} y] = 1$. Thus a simple induction on n, we have $[x, y] = 1$. Lemma 1 implies that G is nilpotent.

Remark *If $G = AB$, A is a normal nilpotent subgroup of G, B is a nilpotent subgroup of G, $(| A |, | B |) = 1$, then G is not necessarily nilpotent. As confirmed by S_3, the symmetric group of degree 3.*

2. Main results

We prove the following theorems:

Theorem 1 *Assume that $G = AB$, that A and B are nilpotent subgroups of G, and that $(| A |, | B |) = 1$. If for each element x in A and each element y in B there is a positive integer n such that $[x,_n y] = 1$, then G is nilpotent.*

Proof Let M be an arbitrary maximal subgroup of G. Since G is solvable, it follows that$| G : M | = p^n$ for some prime p. Since $(| A |, | B |) = 1$, we can assume that p does not divide, say, $| B |$. LetM_1 be a p'-Hall subgroup of M. By π-sylow Theorem [4, Theorem 4.1, P. 231], we have $B \leq M_1^x$ for some x in G. SinceM^x has the same properties as M, we can replace M by M^x and so we can assume without loss of generality that $B \leq M$. Since $G = AB$, it follows that $M = B(A \cap M)$. Clearly, B and $A \cap M$ are nilpotent subgroups of M, $(| B |, | A \cap M |) = 1$. By hypothesis, for each element x in $A \cap M$ and each element y in B there is a positive integer n such that $[x,_n y] = 1$. By induction on$| G |$, M is nilpotent. If G is not nilpotent, then G is a minimal nonnilpotent group. By Ito Theorem [4, Theorem 5.2, P. 281], $G = PQ$, where P is normal in G , and P is a sylow p-subgroup of G, Q is non-normal cyclic sylow q-subgroup of G, $p \neq q$. We can assume that $P = A$ and $Q = B$, so that $A \lhd G$. Due to each element x in A and each element y in B there is a positive integer n such that $[x,_n y] = 1$, Lemma 2 implies that G is nilpotent, a contradiction. This is impossible as G is minimal nonnilpotent group. This completes the proof of the theorem.

Theorem 2 *Assume that $G = AB$, that A is a normal nilpotent subgroup of G, and that B is a nilpotent subgroup of G. If for each element x in A and each element y in B there is a positive integer n such that $[x,_n y] = 1$, then G is nilpotent.*

Proof. Let p be an arbitrary prime dividing $| G |$. We consider the following cases:

Case 1. If $p \mid | A |, p \mid | B |$. Then, by [2, Theorem 27(3), P. 217], there exist a sylow p-subgroupP_1 of A and a sylow p-subgroupP_2 of B such that $P_1 P_2$ is a sylow p-subgroup of G. Let $P = P_1 P_2$, we show that $P \lhd G$. Since B is nilpotent, there exist a p'-Hall subgroup B_2 of B such that $B = P_2 B_2$. If $G = P_1 B$, then$G = PB_2$. By hypothesis , $P_1 \mathrm{char} A \lhd G$, hence $P_1 \lhd G$. Since B is nilpotent, it follows that $P_2 \lhd B$. Hence $P \lhd G$. If $G = P_2 A$, assume that A_1 is a p'-Hall subgroup of A, then $A_1 \mathrm{char} A \lhd G$. $P_2A_1 \leq G$, Lemma 2 implies that P_2A_1 is nilpotent, thus $P = P_1P_2 \lhd G$. Now we can assume that P_1B and P_2A are proper subgroups of G. By induction on $|G|$, P_1B and P_2A are nilpotent. Hence $P \lhd P_1B$ and $P \lhd P_2A$. Since $G = AB$, it follows that $P \lhd G$.

Case 2. If $p \mid\mid A \mid$, but $p \dagger \mid B \mid$. Let P be a sylow p-subgroup of A. Then P is also a sylow p-subgroup of G. By hypothesis, $A \lhd G$, A is a nilpotent subgroup, it follows that $P char A \lhd G$, hence $P \lhd G$.

Case 3. If $p \mid\mid B \mid$, but $p \dagger \mid A \mid$, Let P be sylow p-subgroup of B. Then P is also a sylow p-subgroup of G. By hypothesis, $A \lhd G$, so that $AP = PA$. Lemma 2 implies that AP is nilpotent . Hence $P \lhd AP$. By hypothesis, B is nilpotent. Hence $P \lhd B$. Since $G = AB$, it follows that $P \lhd G$. Since p is an arbitrary prime dividing $|G|$, P is a sylow p-subgroup of G, we can obtain $P \lhd G$. then G is nilpotent. This completes the proof of the theorem.

Theorem 3 *Assume that $N \lhd G$, N and G/N are nilpotent groups. If for each element x in N and each element y in G, there is a positive integer n such that $[x,_n y] = 1$, then G is nilpotent.*

Proof We first prove that there is a nilpotent subgroup A of G such that $G = NA$. Let $\mathcal{F} = \{A \mid A \subseteq G, G = NA\}$. Clearly, $G \in \mathcal{F}$, then $\mathcal{F}$ is a nonempty set. Now let A be a minimal element in the set $\mathcal{F}$, we show that A is nilpotent. Assume that $N \cap A \not\subseteq \Phi(A)$ (Frattini subgroup of A), there exists a maximal subgroup B of A such that $N \cap A \not\subseteq B$. Since $N \cap A \lhd A$, we have $A = (N \cap A)B$. Since $G = NA = N(N \cap A)B = NB$, then $B \in \mathcal{F}$, a contradition. Hence $N \cap A \subseteq \Phi(A)$. Since $G = NA$, so that $G/N \simeq A/N \cap A$. But $A/N \cap A \sim (A/N \cap A)/(\Phi(A)/N \cap A)$, then $G/N \sim (A/N \cap A)/(\Phi(A)/N \cap A) \simeq A/\Phi(A)$. This implies that $G/N \sim A/\Phi(A)$. G/N is nilpotent implies that $A/\Phi(A)$ is nilpotent. Thus A is nilpotent. By hypothesis, $G = NA$, N and A are nilpotent subgroups of G, so for each element x in N and each element y in A there is a positive integer n such that $[x,_n y] = 1$. Theorem 2 implies that G is nilpotent.

Theorem 4 *Assume that G is a finite group that x is an arbitrary element of order p or order $2^2 (p = 2)$, and that y is an arbitrary p'-element of G. If there is positive integer n such that $[x,_n y]$is a p'-element, then G is p-nilpotent.*

Proof Suppose that the consequence is false and let G be a counter-example of the smallest order. Then G is not p-nilpotent group. But each of whose proper subgroup of G is p-nilpotent, by Ito Theorem [4, theorem 5.2, P. 281], $G = PQ$, $\exp(P) = p$ or 2^2. Let$x \in P, y \in Q$. By hypothesis, there is a positive integer n such that $[x,_n y]$ is a p'-element of G. Since $P \lhd G$, we have that $[x,_n y] \in P$, it implies that $[x,_n y] = 1$. By Lemma 2, G is p-nilpotent, a contradiction. This is impossible as G is a counter-example of the smallest order. This completes the proof of the theorem.

Our theorem may be considered as a generalization of the following well-known result:

Theorem 5 *Assume that G is finite group. If each element of G of order p or order $2^2 (p = 2)$ lies in $Z(G)$, then G is p-nilpotent [see 4, Theorem 5.5, P. 435].*

Theorem 6 *Let x be an arbitrary element of G of prime order or order 2^2, y be an arbitrary element of G of prime power order, $(o(x), o(y)) = 1$, if there is a positive integer n such that $[x,_n y] = 1$, then G is nilpotent.*

Proof By Theorem 4, it is obvious.

As an immediate consequence of Theorem 6, we have the following well-known result:

Theorem 7 *Let x and y be arbitrary element of G, if there is a positive integer n such that $[x,_n y] = 1$, then G is nilpotent. [see 4, Theorem 6. 13, P. 447]*

Theorem 8 *Assume that G is finite group, that P is an arbitrary p-subgroup of G, that x is an arbitrary element of P of order p or order 2^2 $(p = 2)$, and that y is an arbitrary p'-element of $N_G(P)$. If there is a positive integer n such that $[x,_n y]$ is a p'-element of G, then G is p-nilpotent.*

Proof Let y be an arbitrary p'-element of $N_G(P)$. We can consider the group $P\langle y\rangle$. Let x be an arbitrary element of $P\langle y\rangle$ of order p or order 2^2. Clearly, $x \in P$, since $P\langle y\rangle \leq N_G(P)$. Thus an arbitrary p'-element in $P\langle y\rangle$ that it is an arbitrary p'-element in $N_G(P)$. We can assume that y' is an arbitrary p'-element in $N_G(P)$. By hypothesis, there is a positive integer n such that $[x,_n y']$ is a p'-element. Theorem 4 implies that $P < y' >$ is p-nilpotent. It follows that $\langle y'\rangle \triangleleft P\langle y'\rangle$. Hence $P\langle y'\rangle = P \times \langle y'\rangle, N_G(P)/C_G(P)$ is p-group. By Frobenius' Theorem, G is p-nilpotent. This completes the proof of the Theorem.

Our Theorem 8 may be considered as a generalization of Theorem 10.24 in [1] [see 1, Theorem 10.24, P. 124].

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关于有限群两个幂零子群积的问题

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摘 要: 众所周知，有限群的两个幂零子群的积不一定是幂零的. 本文研究了 Engel 条件对两个幂零子群的影响，得到两个幂零子群的积为幂零群的几个充分条件.