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L_2-Optimal Recovery on the Riesz Potential Spaces *

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Abstract: The problems of best reconstruction of multivariate functions of the Riesz potential spaces from their values on a given mesh are considered, and the exact results of some classes of $L_2(R^n)$ (and $L_2(Q^n)$) defined by the Riesz potential are obtained.

Key words: Riesz potential; multivariate function; optimal recovery.

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1. Introduction

Let X, Y and W be Banach spaces, V an operator from W to Y, $\mathcal{M}$ a subset of W, and Φ a set of operators from Y to X. In addition, let A be a given operator with domain of definition $\mathcal{D}(\mathcal{A}) \subset \mathcal{W}$, $\mathcal{M} \subset \mathcal{D}(\mathcal{A})$, and range $R(A) \subset X$. We consider the following quantities

$$E_\Phi(A, \mathcal{M}, V, X) =: \inf_{T\in\Phi} \sup_{x\in\mathcal{M}} \|Ax - TVx\|_X, \tag{1.1}$$

$$E(\mathcal{M}, M, X) =: \sup_{x\in\mathcal{M}} \inf_{y\in M} \|x - y\|_X. \tag{1.2}$$

When $\Phi = \Phi_a$ in (1.1) is the set of all operators from Y to X, we simply write $E(\cdot,\cdot,\cdot,\cdot)$ instead of $E_{\Phi_a}(\cdot,\cdot,\cdot,\cdot)$.

For any real $\alpha > 0$, we denote the Riesz potential space of all real functions on R^n by

$$L_2^\alpha(R^n) =: \{f \in L_2(R^n) \cap C(R^n) : \|x\|^\alpha \hat{f}(x) \in L_2(R^n)\}. \tag{1.3}$$

Here $\|x\|$ is the Euclidean norm of the vector $x = (x_1, \cdots, x_n) \in R^n$, and $\hat{f}$ the Fourier transform of the function f in $L_2(R^n)$, namely, when $f \in L_1(R^n) \cap L_2(R^n)$,

$$\mathcal{F}f(x) = \hat{f}(x) = (2\pi)^{-\frac{n}{2}} \int_{R^n} f(t) e^{-it\cdot x} \mathrm{d}t,$$

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while $t \cdot x$ is the Euclidean inner product of the vectors t and x in R^n. If $f \in L_2^\alpha(R^n)$, we denote the function $D^\alpha f$ defined by its Fourier transform $\widehat{D^\alpha f}(x) = \|x\|^\alpha \widehat{f}(x)$. When $\alpha = 2k$ is even, $\Delta^k = (-1)^k D^{2k}$ is the k-iterate of the usual Laplace operator Δ (see [6]). Set

$$W_2^\alpha(R^n) = \{f \in L_2^\alpha(R^n) : \| \|\cdot\|^\alpha \widehat{f}(\cdot)\|_{L_2(R^n)} \le 1\}, \tag{1.4}$$

where $\|g\|_{L_2(I)} =: \left\{ \int_I |g(x)|^2 dx \right\}^{1/2}$ is the L_2-norm on the set $I \subset R^n$. Let $\varphi_\alpha(\|x\|)$, $\alpha > 0$, be the function from R^n to R given by

$$\varphi_\alpha(\|x\|) = C(n,\alpha) \begin{cases} \|x\|^{\alpha-n}, & \text{if } \alpha - n \ne 0, 2, 4, \cdots, \\ \|x\|^{\alpha-n} \log \|x\|, & \text{if } \alpha - n = 0, 2, 4, \cdots, \end{cases} \tag{1.5}$$

where $C(n,\alpha)$ is a constant which depends only on α and n and is chosen so that the generalized Fourier transform (i.e. in the sense of distribution) $\widehat{\varphi}_\alpha(\xi) = (2\pi)^{-n/2}\|\xi\|^{-\alpha}$. Denote by $S_{\alpha,2}$ the space of all real functions f defined by

$$\lim_{N\to\infty} \left\| \sum_{\|\nu\| \le N} c_\nu \varphi_\alpha(\|\cdot - \nu\|) - f \right\|_{L_2(R^n)} = 0, \tag{1.6}$$

where $\{c_\nu;\ \nu \in Z^n\}$ is a subset of R. When $\alpha = 2k$ is even, $S_{\alpha,2}$ is the space of k-polyharmonic cardinal splines (see [6]).

We are interested in the exact values of the quantities $E(D^\gamma, W_2^\alpha(R^n),\ V, L_2(R^n))$ and $E(W_2^\alpha(R^n), S_{\beta,2}, L_2(R^n))$ for $0 < \gamma \le \alpha \le \beta$. Here the operator V is defined by $Vf = \{f(\nu) :\ \nu \in Z^n\}$. When $n = 1$, Sun and Li[12] gave the exact results for the case $\alpha = r \in Z_+$ and $\gamma = 0, 1, \cdots, r$. Further, Chen, Li and Micchilli[2] extended the results of [12] to the case $\alpha \ge 1$. Recently, Yan[14] extended these results to the case $\alpha > \frac{1}{2}$. On the multivariate case, when $\alpha = 2k$ is even, $2k \ge n+1$, $\gamma = 0, 2, \cdots, 2k$, and $\beta = 2m \in Z_+$, the author in [4] gave some exact results. In this paper, we continue to extend these to the case $\alpha > \frac{n}{2}$ and $0 < \gamma \le \alpha \le \beta$.

2. Some results on a periodic class of multivariate functions

In this section, we give some results on a periodic class. Because their proofs are completely similar to those of [4], we omit their details.

For $\alpha > 0$, the multiple trigonometric series

$$\sum_{\nu \in Z^n - \{0\}}^{\infty} \frac{e^{i\nu x}}{\|\nu\|^\alpha},\ x \in R^n,$$

is the Fourier series of an integrable function $D_\alpha(x)$ on Q^n, $Q^n = \{x = (x_1, \cdots, x_n) \in R^n :\ -\pi \le x_j < \pi, j = 1, 2, \cdots, n\}$. For $0 < \alpha < n$, we see this fact from [11, Chapter VII]; for $\alpha > \frac{n}{2}$, it is easy to verify that $D_\alpha \in L_2(Q^n) \subset L_1(Q^n)$. For $1 \le p \le +\infty$, let

$$\widetilde{L}_p^\alpha(Q^n) = \{c + D_\alpha * \phi :\ c \in R, \phi \in L_p(Q^n), \int_{Q^n} \phi(t)\, dt = 0\},$$

and set

$$\widetilde{W}_p^\alpha(Q^n) = \{c + D_\alpha * \phi \in \widetilde{L}_p^\alpha(Q^n) : \|\phi\|_{L_p(Q^n)} \le 1\}.$$

Hence $D_\alpha * \phi$ denote the convolution of D_α and ϕ defined by

$$D_\alpha * \phi(x) = \frac{1}{(2\pi)^n} \int_{Q^n} D_\alpha(x-t)\phi(t)\,\mathrm{d}t.$$

For $f = c + D_\alpha * \phi \in \widetilde{L}_2^\alpha(Q^n)$, we write $D^\alpha f = \phi$. For $N \in Z_+$, we denote by $V_N f$ the information of $f \in C(Q^n)$ defined by $V_N f = \{f(\frac{\nu\pi}{N}) : \nu \in \square_N\}$. Here $\square_N = \{(\nu_1, \cdots, \nu_n) \in Z^n : -N \le \nu_j < N, j = 1, 2, \cdots, n\}$.

In this section, we obtain

Theorem 1 *Let $\beta \ge \alpha > \frac{n}{2}$. Then*

$$E(\widetilde{W}_2^\alpha(Q^n), \widetilde{S}_N^\beta, L_2(Q^n)) = \sup_{f \in \widetilde{W}_2^\alpha(Q^n)} \|s_{N,2\alpha} f - f\|_{L_2(Q^n)} = \frac{1}{N^\alpha}.$$

Here $\widetilde{S}_N^\beta$ =: span$\{1, D_\beta(x + \frac{\nu\pi}{N}) - D_\beta(x) : \nu \in \square_N - \{0\}\}$ and $s_{N,2\alpha} f$ in $\widetilde{S}_N^{2\alpha}$ is the interpolation function which interpolates f at points $\{\frac{\nu\pi}{N} : \nu \in \square_N\}$.

Theorem 2 *Let $\alpha > \frac{n}{2}$ and $\alpha \ge \gamma > 0$. Then*

$$\begin{aligned} E(D^\gamma, \widetilde{W}_2^\alpha(Q^n), V_N, L_2(Q^n)) &= \sup_{f \in \widetilde{W}_2^\alpha(Q^n)} \|D^\gamma s_{N,2\alpha} f - f\|_{L_2(Q^n)} \\ &= \sup\{\|D^\gamma f\|_{L_2(Q^n)} : V_N f = 0, f \in \widetilde{W}_2^\alpha(Q^n)\} = \tfrac{1}{N^{\alpha-\gamma}}, \end{aligned}$$

where $V_N f = 0$ means that $f(\frac{\nu\pi}{N}) = 0$ for all $\nu \in \square_N$.

Remark When $\alpha = 2k$, $(2k \ge n+1)$, $\beta = 2m$ and $\gamma = 2s$ are even, Theorem 1 and Theorem 2 have been obtained by the author [4]. As in [4], the proofs of Theorem 1 and Theorem 2 need the following lemmas.

Lemma 1 *For $\beta > n$, set*

$$\ell_\beta(x) = \frac{1}{(2N)^n} + \frac{1}{(2N)^n} \sum_{\nu \in \square_N - \{0\}} A_\nu^{-1} \sum_{j \in Z^n} \frac{e^{i(2jN-\nu)x}}{\|2jN - \nu\|^\beta},$$

where $A_\nu = \sum_{j \in Z^n} \|2jN - \nu\|^{-\beta}, \nu \in \square_N - \{0\}$. Then,

(i) $\ell_\beta(\frac{j\pi}{N}) = \delta_{0j}$, $j \in \square_N$, where δ_{0j} is the Kronecker delta,

(ii) $\ell_\beta(x) = \frac{1}{(2N)^n} + \frac{1}{(2N)^n} \sum_{\nu \in \square_N} d_\nu D_\beta(x + \frac{\nu\pi}{N})$,

where $d_\nu =: \frac{1}{(2N)^n} \sum_{u \in \square_N \setminus \{0\}} A_u^{-1} e^{\frac{i\pi u \cdot \nu}{N}}, \nu \in \square_N$, satisfying $\sum\limits_{\nu \in \square_N} d_\nu = 0$, namely, $\ell_\beta(x)$ is the fundamental function of interpolation for the function set $\widetilde{S}_N^\beta$.

For $\beta > n$, and a bounded function f, we define the interpolation operator $s_{N,\beta}$ by

$$s_{N,\beta} f(x) = \sum_{\nu \in \square_N} f\Big(\frac{\nu\pi}{N}\Big) \ell_\beta\Big(x - \frac{\nu\pi}{N}\Big).$$

Lemma 2 *Let $\alpha > \frac{n}{2}$. Then*

$$\|D^\alpha f - D^\alpha(s_{N,2\alpha}f)\|^2_{L_2(Q^n)} = \|D^\alpha f\|^2_{L_2(Q^n)} - \|D^\alpha(s_{N,2\alpha}f)\|^2_{L_2(Q^n)},$$

for all $f \in \widetilde{L}^\alpha_2(Q^n)$.

Remark When $\alpha = 2k$ is even and $2k \geq n+1$, Lemma 2 may be seen in [4], but there the power 2 was missed.

Lemma 3 *Let $\alpha > \frac{n}{2}$ and $0 < \gamma < \alpha$. Then*

$$\|D^\gamma f\|_{L_2(Q^n)} \leq \|D^\alpha f\|^{\gamma/\alpha}_{L_2(Q^n)} \|f\|^{1-\frac{\gamma}{\alpha}}_{L_2(Q^n)},$$

for all $f \in \widetilde{L}^\alpha_2(Q^n)$.

Lemma 4 *For $\alpha > \frac{n}{2}$ and $0 < \gamma < \alpha$, set*

$$D(D^\gamma, \widetilde{W}^\alpha_2(Q^n), V_N, L_2(Q^n)) = \sup\{\|D^\gamma f\|_{L_2(Q^n)} : \ f \in \widetilde{W}^\alpha_2(Q^n),\ V_N f = 0\}.$$

Then, we have

$$D(D^\gamma, \widetilde{W}^\alpha_2(Q^n), V_N, L_2(Q^n)) = \frac{1}{N^{\alpha-\gamma}}.$$

3. Some results on a class of multivariate functions defined on R^n

Let $\varphi(R^n)$ and $\varphi'(R^n)$ denote the Schwartz space of rapidly desreasing functions and its dual, the space of tempered distributions. To state our results, we need the cardinal interpolation of the radical basis function $\varphi_\alpha(\|x\|)$. As in [1], if $\alpha > n$, then it is easy to verify that $\varphi_\alpha(\|x\|)$ is admissible of order m, where we have put $\alpha = m + n - \mu$ for $\mu \in [0,1)$ and $m \in Z$. (see definition 4 in [1] for the sense of admissible). Therefore, from [1], we have

Lemma 5 *For $\alpha > n$, there exists a cardinal function $\chi_\alpha(x)$ defined by*

$$\chi_\alpha(x) = \frac{1}{(2\pi)^n}\int_{R^n} \frac{\widehat{\varphi}_\alpha(t)e^{itx}}{\sum_{\ell\in Z^n}\widehat{\varphi}_\alpha(t+2\pi\ell)}\,dt \tag{3.1}$$

such that

(1) $\chi_\alpha(x)$ has the representation

$$\chi_\alpha(x) = \sum_{\nu\in Z^n} c_\nu \varphi_\alpha(\|x-\nu\|) \tag{3.2}$$

which the right series is an absolutely convergent sum;

(2) $\chi_\alpha(j) = \delta_{0,j}$ for all $j \in Z^n$;

(3) $\chi_\alpha(x) \in C(R^n)$ and $\chi_\alpha(x) = O(\|x\|^{-\alpha-n})$ as $\|x\| \to \infty$.

Lemma 5 is a corollary of Theorem 6, 10 and 11 of [1]. For $\alpha > n$, $0 < \gamma < \alpha$, and a locally bounded function f satisfying $|f(x)| = O(\|x\|^\gamma)$ as $\|x\| \to \infty$, set

$$(S_\alpha f)(x) = \sum_{\nu \in Z^n} f(\nu)\chi_\alpha(x-\nu), \tag{3.3}$$

where the right series of (3.3) is an absolutely convergent sum on R^n.

In this section, we obtain

Theorem 3 *Let $\alpha > \frac{n}{2}$, $0 < \gamma < \alpha$. Then, we have*

$$\begin{aligned} E(D^\gamma, W_2^\alpha(R^n), V, L_2(R^n)) &= \sup\{\|D^\gamma f\|_{L_2(R^n)} : \ Vf = 0, \ f \in W_2^\alpha(R^n)\} \\ &= \sup_{f \in W_2^\alpha(R^n)} \|D^\gamma f - D^\gamma S_{2\alpha} f\|_{L_2(R^n)} = \tfrac{1}{\pi^{\alpha-\gamma}}. \end{aligned} \tag{3.4}$$

Theorem 4 *Let $\beta \geq \alpha > \frac{n}{2}$ and $\beta > n$. Then, we have*

$$E(W_2^\alpha(R^n), S_{\beta,2}, L_2(R^n)) = \sup_{f \in W_2^\alpha(R^n)} \|f - S_{2\alpha} f\|_{L_2(R^n)} = \frac{1}{\pi^\alpha}. \tag{3.5}$$

To prove Theorem 3 and 4, we first give the following lemmas.

Lemma 6 *Let $\alpha > \gamma > 0$. Then the inequality*

$$\|D^\gamma f\|_{L_2(R^n)} \leq \|D^\alpha f\|_{L_2(R^n)}^{\gamma/\alpha} \|f\|_{L_2(R^n)}^{1-\gamma/\alpha} \tag{3.6}$$

holds for any $f \in L_2^\alpha(R^n)$.

The proof of Lemma 6 is similar to that of Lemma 7 of [4] for the case $\alpha = 2k$, $\gamma = 2s$, and its detail is omitted.

Lemma 7[8] *Schwartz space $\varphi(R^n)$ is dense in $L_2^\alpha(R^n)$.*

Lemma 8 *Let $f \in L_2^\alpha(R^n)$ for $\alpha > \frac{n}{2}$. Then $\widehat{f}$ belongs to $L_1(R^n)$ and hence the function f is uniformly continuous and bounded on R^n.*

Proof Since $f \in L_2^\alpha(R^n)$, then $(1 + \|x\|^\alpha)\widehat{f}(x) \in L_2(R^n)$. Hence, note that $\frac{1}{1+\|x\|^\alpha} \in L_2(R^n)$ for $\alpha > \frac{n}{2}$, we have $\widehat{f}(x) = \frac{1}{1+\|x\|^\alpha} \cdot (1 + \|x\|^\alpha)\widehat{f}(x) \in L_1(R^n)$.

Lemma 9 *If $\alpha > \frac{n}{2}$ and $f \in L_2^\alpha(R^n)$, then $S_{2\alpha} f \in L_2^\alpha(R^n)$.*

Proof As in [7], the proof is divided into two steps.

(1) Let $f \in \varphi(R^n)$. Recall that

$$\widehat{S_{2\alpha} f}(\xi) = (2\pi)^{n/2} U(\xi) \widehat{\chi}_{2\alpha}(\xi), \tag{3.7}$$

where, in view of Poisson formula

$$U(\xi) = (2\pi)^{-n/2} \sum_{j \in Z^n} f(j) e^{-ij\cdot\xi} = \sum_{j \in Z^n} \widehat{f}(\xi - 2\pi j).$$

We now prove that

$$\int_{R^n}(1+\|\xi\|^2)^\alpha|U(\xi)\widehat{S}_{2\alpha}(\xi)|^2\,\mathrm{d}\xi \le C\int_{R^n}(1+\|\xi\|^2)^\alpha|\widehat{f}(\xi)|^2\,\mathrm{d}\xi, \tag{3.8}$$

where the constant C is independent of f.

Let a_j be the maximum of $(1+\|\xi\|^2)^{-\alpha}(1+\|\xi-2j\pi\|^2)^\alpha\cdot|\widehat{\chi}_{2\alpha}(\xi-2j\pi)|^2$, for ξ in Q^n, and note that $|\widehat{\chi}_{2\alpha}(\xi)|\le a_0=1$, for ξ in Q^n. Since $a_j=O(\|j\|^{-2\alpha})$, for large $\|j\|$, we may write

$$\begin{aligned}\textstyle\int_{R^n}\ &(1+\|\xi\|^2)^\alpha|U(\xi)\widehat{\chi}_{2\alpha}(\xi)|^2\,\mathrm{d}\xi\\ &=\textstyle\sum_{j\in Z^n}\int_{Q^n}|U(\xi)|^2(1+\|\xi-2j\pi\|^2)^\alpha|\widehat{\chi}_{2\alpha}(\xi-2j\pi)|^2\,\mathrm{d}\xi\\ &\le\textstyle\left(\sum_{j\in Z^n}a_j\right)\int_{Q^n}(1+\|\xi\|^2)^\alpha|U(\xi)|^2\,\mathrm{d}\xi=A\int_{Q^n}(1+\|\xi\|^2)^\alpha|U(\xi)|^2\,\mathrm{d}\xi,\end{aligned}\tag{3.9}$$

where we have put $A=\sum_{j\in Z^n}a_j$.

Observe that for ξ in Q^n we may write

$$\begin{aligned}(1+\|\xi\|^2)^\alpha|U(\xi)|^2&\le\textstyle\left(\sum_{j\in Z^n}(1+\|\xi\|^2)^{\alpha/2}|\widehat{f}(\xi-2j\pi)|\right)^2\\ &\le\textstyle\left(\sum_{j\in Z^n}b_j(1+\|\xi-2j\pi\|^2)^{\alpha/2}|\widehat{f}(\xi-2j\pi)|\right)^2\\ &=\textstyle\sum_{j\in Z^n}\sum_{\nu\in Z^n}b_jb_\nu(1+\|\xi-2\nu\pi\|^2)^{\alpha/2}|\widehat{f}(\xi-2\nu\pi)|(1+\|\xi-2j\pi\|^2)^{\alpha/2}|\widehat{f}(\xi-2j\pi)|,\end{aligned}$$

where $b_0=1$ and otherwise b_j is equal to the maximum of $((\|\xi\|^2+1)^{\alpha/2}(1+\|\xi-2j\pi\|^2)^{-\alpha/2}$ over ξ in Q^n. Integrating the last expression involving U over Q^n and observing that

$$\int_{Q^n}(1+\|\xi-2\nu\pi\|^2)^{\alpha/2}|\widehat{f}(\xi-2\nu\pi)|(1+\|\xi-2j\pi\|^2)^{\alpha/2}|\widehat{f}(\xi-2j\pi)|\,\mathrm{d}\xi\le V_jV_\nu,$$

where we have put $V_j=\left(\int_{Q^n}(1+\|\xi-2j\pi\|^2)^\alpha|\widehat{f}(\xi-2j\pi)|^2\,\mathrm{d}\xi\right)^{1/2}$. Hence, we have

$$\int_{Q^n}(1+\|\xi\|^2)^\alpha|U(\xi)|^2\,\mathrm{d}\xi\le\sum_{\nu\in Z^n}\sum_{j\in Z^n}b_\nu b_jV_jV_\nu=\Big(\sum_{j\in Z^n}b_jV_j\Big)^2.$$

Note that $2\alpha>n$ implies that the sum $\sum\limits_{j\in Z^n}b_j^2$ is finite. Then by Schwartz's inequality we have

$$\int_{Q^n}(1+\|\xi\|^2)^\alpha|U(\xi)|^2\,\mathrm{d}\xi\le\Big(\sum_{j\in Z^n}b_j^2\Big)\Big(\sum_{j\in Z^n}V_j^2\Big).\tag{3.10}$$

Since $\sum\limits_{j\in Z^n}V_j^2=\int_{R^n}(1+\|\xi\|^2)^\alpha|\widehat{f}(\xi)|^2\,\mathrm{d}\xi$, then by (3.7), (3.9) and (3.10) we have

$$\int_{R^n}(1+\|\xi\|^2)^\alpha|\widehat{S_{2\alpha}f}(\xi)|^2\,\mathrm{d}\xi\le C\int_{R^n}(1+\|\xi\|^2)^\alpha|\widehat{f}(\xi)|^2\,\mathrm{d}\xi,\tag{3.11}$$

for all $f \in \varphi(R^n)$ and $C =: \sum_{j \in Z^n} b_j^2$.

(2) Let $f \in L_2^\alpha(R^n)$. By Lemma 7, we take f_N in $\varphi(R^n)$ for $N \in Z_+$, such that

$$\lim_{N\to\infty} \|(1+\|\ \cdot\ \|^2)^{\alpha/2}|\widehat{f}_N - \widehat{f}|\,\|_{L_2(R^n)} = 0. \tag{3.12}$$

Note that $\alpha > \frac{n}{2}$ and

$$|f_N(x) - f(x)| \le (2\pi)^{-n/2}\Big\|\frac{1}{(1+\|\cdot\|^2)^{\alpha/2}}\Big\|_{L_2(R^n)}\|(1+\|\cdot\|^2)^{\alpha/2}|\widehat{f}_N - \widehat{f}|\|_{L_2(R^n)}.$$

Then, by (3.12), $f_N(x)$ converges uniformly to $f(x)$ on R^n as $N \to \infty$, and by (3.11), $\{(1+\|x\|^2)^{\alpha/2}\widehat{S_{2\alpha}f_N}\}$ is a Cauchy sequence in $L_2(R^n)$. Let $\widehat{g}(x)$ be the $L_2(R^n)$–limit of $(1+\|x\|^2)^{\alpha/2}\widehat{S_{2\alpha}f_N}(x)$. Then, for any $h \in \varphi(R^n)$, we have

$$\int_{R^n}\widehat{g}(x)h(x)\,\mathrm{d}x = \lim_{N\to\infty}\int_{R^n}(1+\|x\|^2)^{\alpha/2}\widehat{S_{2\alpha}f_N}(x)h(x)\,\mathrm{d}x$$

$$= \lim_{N\to\infty}\int_{R^n} S_{2\alpha}f_N(x)\cdot\mathcal{F}((1+\|\ \cdot\ \|^2)^{\alpha/2}h)(x)\,\mathrm{d}x$$

$$= \int_{R^n} S_{2\alpha}f(x)\mathcal{F}((1+\|\ \cdot\ \|^2)^{\alpha/2}h)(x)\,\mathrm{d}x$$

$$= \int_{R^n}(1+\|x\|^2)^{\alpha/2}\widehat{S_{2\alpha}f}(x)h(x)\,\mathrm{d}x$$

which shows that $\widehat{g}(x) = (1+\|x\|^2)^{\alpha/2}\widehat{S_{2\alpha}f}(x)$, $x \in R^n$, i.e. $S_{2\alpha}f \in L_2^\alpha(R^n)$.

As in [8] and [9], we denote the difference operator $\Delta_t^\ell f$ defined by

$$(\Delta_t^\ell f)(x) = \sum_{k=0}^{\ell}(-1)^k C_\ell^k f(x-kt), \quad t \in R^n,$$

where ℓ is a positive integer and $C_\ell^k = \frac{\ell!}{k!(\ell-k)!}$, $k = 0,1,\cdots,\ell$. Then, the differential operator D^α of fraction order $\alpha > 0$ is given by $(D^\alpha f)(x) = L_2(R^n) - \lim_{\varepsilon\to 0^+}(D_\varepsilon^\alpha f)(x)$, while $(D_\varepsilon^\alpha f)(x) = \frac{1}{d_{n,\ell}(\alpha)}\int_{\|t\|>\varepsilon}\frac{(\Delta_t^\ell f)(x)}{\|t\|^{n+\alpha}}\,\mathrm{d}t$, for $\ell > \alpha$. Here the constant $d_{n,\ell}(\alpha)$ is chosen so that $(\widehat{D^\alpha f})(x) = \|x\|^\alpha\widehat{f}(x)$.

Lemma 10 *Let the function $\mu \in C^\infty(R^n)$ satisfy the properties: supp $\mu \subset [-2,2]^n$; $|\mu(x)| \le 1$, for $x \in R^n$; $\mu(x) \equiv 1$ for $x \in [-1,1]^n$. For any $f \in L_2^\alpha(R^n)$, $\alpha > \frac{n}{2}$, if we define $f_N(x) = \mu(\frac{x}{N})f(x)$, then*

$$\lim_{N\to\infty}\|D^\gamma f_N - D^\gamma f\|_{L_2(R^n)} = 0$$

holds for $\gamma \in [0,\alpha]$.

Proof Since $\|\mu(\frac{\cdot}{N})D^\gamma f - D^\gamma f\|_{L_2(R^n)} \to 0$ as $N \to \infty$, it is sufficient to verify that

$$\|D^\gamma f_N - \mu(\frac{\cdot}{N})D^\gamma f\|_{L_2(R^n)} \to 0, \quad \text{as } N \to \infty, \tag{3.13}$$

for $\gamma \in [0,\alpha]$. We prove the fact only for $\gamma = \alpha$, the other cases are similar.

As in [8], if $a(t)$ is a function, by the formula

$$(\Delta_t^\ell a\cdot f)(x)=\sum_{k=0}^{\ell}C_\ell^k(\Delta_t^k a)(x)(\Delta_t^{\ell-k}f)(x-kt),$$

we see that

$$(D^\alpha f_N)(x)=\mu\Big(\frac{x}{N}\Big)D^\alpha f(x)+\tfrac{1}{d_{n,\ell}(\alpha)}\textstyle\sum_{k=1}^{\ell}C_\ell^k B_{N,k}f(x),$$

$$B_{N,k}f(x)=\int_{R^n}\frac{(\Delta_t^k\mu(\frac{\cdot}{N}))(x)(\Delta_t^{\ell-k}f)(x-kt)}{\|t\|^{n+\alpha}}\,\mathrm{d}t.$$

Next, we prove that

$$\lim_{N\to\infty}\|B_{N,k}f\|_{L_2(R^n)}=0,\ \text{for } k=1,2,\cdots,\ell. \tag{3.14}$$

For $k=1,2,\cdots,\ell$, note that $|(\Delta_t^k\mu(\frac{\cdot}{N}))(x)|\le C\Big\|\frac{t}{N}\Big\|^k\Big(1+\Big\|\frac{t}{N}\Big\|\Big)^{-k}$ for some constant C. Then

$$\begin{aligned}|B_{N,k}f(x)|&\le \tfrac{C}{N^k}\int_{R^n}\frac{|\Delta_t^{\ell-k}f(x-kt)|}{\|t\|^{n+\alpha-k}\Big(1+\frac{\|t\|}{N}\Big)^k}\,\mathrm{d}t\\ &\le \tfrac{C}{N^k}\int_{\|t\|\le1}\frac{|\Delta_t^{\ell-k}f(x-kt)|}{\|t\|^{n+\alpha-k}}\,\mathrm{d}t+C\textstyle\sum_{\nu=0}^{\ell-k}C_{\ell-k}^{\nu}\int_{\|t\|\ge1}\frac{|f(x-(k+\nu)t)|}{\|t\|^{n+\alpha-k}(N+\|t\|)^k}\,\mathrm{d}t.\end{aligned} \tag{3.15}$$

Since $f\in L_2^\alpha(R^n)$, it is easy to verify that

$$\|\Delta_t^{\ell-k}f(\cdot-kt)\|_{L_2(R^n)}=O(\|t\|^\theta),$$

where $\theta=\min\{\ell-k,\alpha\}$.

Hence, we have

$$\Big\|\int_{\|t\|\le1}\frac{|\Delta_t^{\ell-k}f(\cdot-kt)|}{\|t\|^{n+\alpha-k}}\,\mathrm{d}t\Big\|_{L_2(R^n)}=O\Big(\int_{\|t\|\le1}\|t\|^{-n-\alpha+k+\theta}d\theta\Big)=O(1), \tag{3,16}$$

as $N\to\infty$,

$$\Big\|\int_{\|t\|\ge1}\frac{f(\cdot-(k+\nu)t)}{\|t\|^{n+\alpha-k}(N+\|t\|)^k}\,\mathrm{d}t\Big\|_{L_2(R^n)}\le\|f\|_{L_2(R^n)}\int_{\|t\|\ge1}\frac{dt}{\|t\|^{n+\alpha-k}(N+\|t\|)^k}=o(1), \tag{3.17}$$

as $N\to\infty$. By (3.15) to (3.17), we have

$$\|B_{N,k}f\|_{L_2(R^n)}=O\Big(\frac{1}{N^k}\Big)+C\sum_{\nu=0}^{\ell-k}C_{\ell-k}^{\nu}o(1)=o(1),\ (N\to\infty).$$

Therefore, (3.14) is valid. Further, Lemma 10 follows (3.13) and (3.14).

Lemma 11 *If $\alpha>\frac{n}{2}$, then the equality*

$$\|D^\alpha f-D^\alpha S_{2\alpha}f\|_{L_2(R^n)}^2=\|D^\alpha f\|_{L_2(R^n)}^2-\|D^\alpha S_{2\alpha}f\|_{L_2(R^n)}^2$$

holds for any $f \in L_2^\alpha(R^n)$.

Remark When $\alpha = 2k$ is even and $2k \geq n+1$, Lemma 11 may be seen in [7], but there the power 2 is missed.

Proof of Lemma 11 It is sufficient to verify that

$$\int_{R^n} D^\alpha(S_{2\alpha}f)(x) D^\alpha g(x)\,\mathrm{d}x = 0 \tag{3.18}$$

holds for any $g \in L_2^\alpha(R^n)$ with $Vg = 0$.

Let $\mu(t) \in C^\infty(R^n)$ be defined in Lemma 10 and set $f_N(x) = \mu(\frac{x}{N})f(x)$ and $g_N(x) = \mu(\frac{x}{N})g(x)$. Then, by Lemma 10, we have

$$\lim_{N\to\infty} \|D^\alpha S_{2\alpha} f_N - D^\alpha S_{2\alpha} f\|_{L_2(R^n)} = 0, \tag{3.19}$$

$$\lim_{N\to\infty} \|D^\alpha g_N - D^\alpha g\|_{L_2(R^n)} = 0. \tag{3.20}$$

Let c_j be the Fourier coefficients of the periodic function $\|x\|^{2\alpha}\widehat{\chi}_{2\alpha}(x)$ and note that $c_j = O(\|j\|^{-n-\alpha})$ for large $\|j\|$ (see [1]). Then by (3.19), (3.20) and Lemma 8, we have

$$\begin{aligned}
&\textstyle\int_{R^n} D^\alpha S_{2\alpha} f(x) D^\alpha g(x)\,\mathrm{d}x = \lim_{N\to\infty} \int_{R^n} D^\alpha S_{2\alpha} f_N(x) D^\alpha g_N(x)\,\mathrm{d}x \\
&= \textstyle\lim_{N\to\infty} \int_{R^n} \left(\sum_{\nu\in Z^n} f_N(\nu) e^{i\nu x}\right) \|x\|^{2\alpha} \widehat{\chi}_{2\alpha}(x) \widehat{g}_N(x)\,\mathrm{d}x \\
&= \textstyle\lim_{N\to\infty} \sum_{\nu\in Z^n} \sum_{j\in Z^n} c_j f_N(\nu) g_N(\nu + j) = 0,
\end{aligned}$$

in views of $\widehat{g}_N \in L_1(R^n)$ and $Vg_N = 0$.

Proof of Theorem 3 If $f \in W_2^\alpha(R^n)$, Lemma 11 implies that the function $g =: f - S_{2\alpha}f$ belongs to $W_2^\alpha(R^n)$. Note that $Vg = 0$. We have

$$\begin{aligned}
E(D^\gamma, W_2^\alpha(R^n), V, L_2(R^n)) &\leq \textstyle\sup_{f\in W_2^\alpha(R^n)} \|D^\gamma f - D^\gamma S_{2\alpha} f\|_{L_2(R^n)} \\
&\leq \sup\{\|D^\gamma f\|_{L_2(R^n)} :\ Vf = 0,\ f \in W_2^\alpha(R^n)\}.
\end{aligned} \tag{3.21}$$

First we prove that

$$E(D^\gamma, W_2^\alpha(R^n), V, L_2(R^n)) \geq \frac{1}{\pi^{\alpha-\gamma}}, \tag{3.22}$$

for $\gamma \in [0, \alpha]$.

As in [2], let the function μ be defined in Lemma 10, and set

$$f_N(x) = \mu\left(\frac{x}{N}\right) \sin \pi x_1, \text{ for } x = (x_1, x_2, \cdots, x_n) \in R^n.$$

Then $\widehat{f}_N(x) = \frac{N^n}{2i}[\widehat{\mu}(N(x - e_1\pi)) - \widehat{\mu}(N(x + e_1\pi))]$, where we have put $e_1 = (1, 0, \cdots, 0) \in R^n$. By Plancherel formula we have

$$\begin{aligned}
\|D^\gamma f_N\|^2_{L_2(R^n)} &= \textstyle\int_{R^n} \|\xi\|^{2\gamma} |\widehat{f}_N(\xi)|^2\,\mathrm{d}\xi \\
&= \textstyle\frac{N^{2n}}{4} \int_{R^n} \|\xi\|^{2\gamma} |\widehat{\mu}(N(\xi - e_1\pi)) - \widehat{\mu}(N(\xi + e_1\pi))|^2\,\mathrm{d}\xi.
\end{aligned} \tag{3.23}$$

Note that

$$2^{n-1}N^n \le \|\widehat{f}_N\|^2_{L_2(R^n)} = \|f_N\|^2_{L_2(R^n)} \le 2(4^{n-1}N^n).$$

Then, $\int_{R^n} |\widehat{\mu}(N(\xi - e_1\pi)) - \widehat{\mu}(N(\xi + e_1\pi))|^2 \, d\xi \ge \frac{2^n}{N^n}$, and

$$\begin{aligned} &\left| \frac{\|D^\gamma f_N\|^2_{L_2(R^n)}}{\|f_N\|^2_{L_2(R^n)}} - \pi^{2\gamma} \right| = \frac{\int_{R^n} (\|\xi\|^{2\gamma} - \pi^{2\gamma}) |\widehat{\mu}(N(\xi - e_1\pi)) - \widehat{\mu}(N(\xi + e_1\pi))|^2 \, d\xi}{\int_{R^n} |\widehat{\mu}(N(\xi - e_1\pi)) - \widehat{\mu}(N(\xi + e_1\pi))|^2 \, d\xi} \\ &\le \varepsilon + \left(\frac{N}{2}\right)^n \int_{E_\delta} (\|\xi\|^{2\gamma} + \pi^{2\gamma}) |\widehat{\mu}(N(\xi - e_1\pi)) - \widehat{\mu}(N(\xi + e_1\pi))|^2 \, d\xi, \end{aligned} \tag{3.24}$$

where $E_\delta =: \{\xi \in R^n : \ \|\xi\| > \pi + \delta \text{ or } \|\xi\| < \pi - \delta\}$, while the $\delta > 0$ is chosen so that

$$\left| \|\xi\|^{2\gamma} - \pi^{2\gamma} \right| < \varepsilon, \text{ for } \xi \in R^n \text{ with } \|\xi\| \in (\pi - \delta, \pi + \delta).$$

Since $\mu \in C^\infty(R^n)$ and $\operatorname{supp} \mu \in [-2,2]^n$, we have

$$\widehat{\mu}(N\xi) = \frac{1}{(2\pi)^{n/2}} \int_{[-2,2]^n} \mu(x) e^{-ix\cdot N\xi} \, dx = O((N\|\xi\|)^{-2M}), \ \|\xi\| \ne 0, \tag{3.25}$$

for large N, where M is any positive integer. By taking the integer $M > n + \gamma$, we have

$$\lim_{N\to\infty} \frac{\|D^\gamma f_N\|^2_{L_2(R^n)}}{\|f_N\|^2_{L_2(R^n)}} = \pi^{2\gamma},$$

for all $\gamma \in [0, \alpha]$ and finally conclude that

$$\lim_{N\to+\infty} \frac{\|D^\gamma f_N\|^2_{L_2(R^n)}}{\|D^\alpha f_N\|^2_{L_2(R^n)}} = \lim_{N\to+\infty} \frac{\|D^\gamma f_N\|^2_{L_2(R^n)}}{\|f_N\|^2_{L_2(R^n)}} \cdot \frac{\|f_N\|^2_{L_2(R^n)}}{\|D^\alpha f_N\|^2_{L_2(R^n)}} = \pi^{2(\gamma-\alpha)}. \tag{3.26}$$

Let $F_N(x) = \frac{f_N(x)}{\|D^\gamma f_N\|_{L_2(R^n)}}$. Then, note that $VF_N(\,\cdot\,) = 0$ and $VF_N(-\cdot\,) = 0$, we have

$$\begin{aligned} &E(D^\gamma, W_2^\alpha(R^n), V, L_2(R^n)) \ge \inf\nolimits_{T\in\Phi^a} \max\{\|D^\gamma F_N(\,\cdot\,) - TVF_N(\,\cdot\,)\|_{L_2(R^n)}, \\ &\|D^\gamma F_N(-\,\cdot\,) - TVF_N(-\,\cdot\,)\|_{L_2(R^n)}\} \ge \|D^\gamma F_N\|_{L_2(R^n)}. \end{aligned} \tag{3.27}$$

By (3.26), we have

$$\lim_{N\to\infty} \|D^\gamma F_N\|_{L_2(R^n)} = \lim_{N\to\infty} \frac{\|D^\gamma f_N\|_{L_2(R^n)}}{\|D^\alpha f_N\|_{L_2(R^n)}} = \pi^{-(\alpha-\gamma)}. \tag{3.28}$$

Thus, by (3.27) and (3.28) we obtain (3.22).

Next, we prove that the inequality

$$\sup\{\|D^\gamma f\|_{L_2(R^n)} : \ f \in W_2^\alpha(R^n), Vf = 0\} = \pi^{-(\alpha-\gamma)} \tag{3.29}$$

holds for any $\gamma \in [0, \alpha]$. For any $f \in W_2^\alpha(R^n)$ with $Vf = 0$ and $\|D^\alpha f\|_{L_2(R^n)} = 1$, set $f_N(x) = \mu(\frac{x}{N}) f(x)$ and $F_N(x) = f_N(\frac{2N}{\pi} x)$. Let $\widetilde{F}_N$ denote the extension of period 2π

of F_N in each variable, i.e. $\widetilde{F}_N(x + 2\nu\pi) = F_N(x)$ for any x in Q^n and $\nu \in Z^n$. Then $V_{2N}\widetilde{F}_N = 0$. By Theorem 2, we have

$$\|D^\gamma \widetilde{F}_N\|_{L_2(Q^n)} \le \left(\frac{1}{2N}\right)^{\alpha-\gamma} \|D^\alpha \widetilde{F}_N\|_{L_2(Q^n)}. \tag{3.30}$$

By a change of variable, we have

$$1 \ge \frac{(2N)^{-\gamma}\|D^\gamma \widetilde{F}_N\|_{L_2(Q^n)}}{(2N)^{-\alpha}\|D^\alpha \widetilde{F}_N\|_{L_2(Q^n)}} = \pi^{\alpha-\gamma} \cdot \frac{\|D^\gamma f_N\|_{L_2(R^n)}}{\|D^\alpha f_N\|_{L_2(R^n)}}.$$

Note that $\lim\limits_{N\to\infty} \|D^\alpha f_N\|_{L_2(R^n)} = 1$. By Lemma 10, we have

$$\|D^\gamma f\|_{L_2(R^n)} = \lim_{N\to+\infty} \frac{\|D^\gamma f_N\|_{L_2(R^n)}}{\|D^\alpha f_N\|_{L_2(R^n)}} \le \frac{1}{\pi^{\alpha-\gamma}}. \tag{3.31}$$

for any $f \in W_2^\alpha(R^n)$ with $Vf = 0$.

Thus, we obtain (3.29). Proof of Theorem 3 is complete.

To prove Theorem 4, we need the following lemmas.

Lemma 12 *For $\beta > n$, set*

$$K_\beta(x,t) = \frac{e^{ixt} \sum\limits_{j\in Z^n} \frac{1-e^{2\pi ijx}}{\|t+2\pi j\|^\beta}}{\|t\|^\beta \sum\limits_{j\in Z^n} \frac{1}{\|t+2\pi j\|^\beta}}. \tag{3.32}$$

Then, we have

$$K_\beta(x,t) = \frac{e^{ixt} - \sum\limits_{\nu\in Z^n} e^{i\nu t}\chi_\beta(x-\nu)}{\|t\|^\beta}. \tag{3.33}$$

Proof As in [5], for each $\xi \in R^n$, set

$$g(x) = e^{-ix\xi} \sum_{\nu\in Z^n} e^{i\nu\xi}\chi_\beta(x-\nu). \tag{3.34}$$

Then, g is a periodic function, i.e., $g(x+2\pi\nu) = g(x)$ for any $x \in R^n$ and $\nu \in Z^n$. Further, it is easy to verify that g has the Fourier series expression

$$g(x) = \sum_{j\in Z^n} \left(\frac{\|\xi + 2\pi j\|^{-\beta}}{\sum\limits_{\nu\in Z^n} \|\xi + 2\pi\nu\|^{-\beta}}\right) e^{2\pi ij\cdot x}, \quad x \in R^n. \tag{3.35}$$

Thus, by (3.34) and (3.35) we obtain (3.33).

Lemma 13 *For $\beta > n$, let $G_\beta(x,t) = \frac{1}{(2\pi)^n}\int_{R^n} K_\beta(x,\xi)e^{-i\xi t}\,\mathrm{d}\xi$. Then, we have*

(i) $G_\beta(x,t) = G_\beta(t,x)$ for all $x \in R^n$, $t \in R^n$;

(ii) $G_\beta(x,t) = \varphi_\beta(x-t) - \sum_{\nu\in Z^n} \varphi_\beta(\nu - t)\chi_\beta(x-\nu)$;

(iii) $\int_{R^n} |G_\beta(x,t)|\mathrm{d}t < \infty$;

(iv) If $f \in L_2^\beta(R^n)$, then $f(x) - (S_\beta f)(x) = \int_{R^n} G_\beta(x,t) D^\beta f(t)\,dt$.

Proqf First, by the definition of $G_\beta(x,t)$, we have

$$\begin{aligned}(2\pi)^n G_\beta(x,t) &= \int_{R^n} \frac{e^{ix\xi} \sum\limits_{\nu\in Z^n} \frac{1-e^{2\pi i\nu x}}{\|\xi+2\pi\nu\|^\beta}}{\|\xi\|^\beta \sum\limits_{\nu\in Z^n} \|\xi+2\pi\nu\|^{-\beta}} e^{-i\xi t}\,d\xi \\ &= \sum_{j\in Z^n} \int_{Q^n+2j\pi} \frac{\sum\limits_{\nu\in Z^n} \frac{1-e^{2\pi i\nu x}}{\|\xi+2\pi\nu\|^\beta}}{\|\xi\|^\beta \sum\limits_{\nu\in Z^n} \|\xi+2\pi\nu\|^{-\beta}} e^{i(x-t)\xi}\,d\xi \\ &= \int_{Q^n} \frac{E(x-t,\xi)E(0,\xi) - E(-t,\xi)E(x,\xi)}{\sum\limits_{\nu\in Z^n} \|\xi+2\pi\nu\|^{-\beta}} e^{i(x-t)\xi}\,d\xi.\end{aligned}$$

Here we have put

$$E(x,\xi) = \sum_{\nu\in Z^n} \frac{e^{2\pi i\nu x}}{\|\xi+2\nu\pi\|^\beta}.$$

Using the facts that $E(-t,\xi) = E(t,-\xi)$ and $E(0,\xi) = E(0,-\xi)$, we see that $G_\beta(x,t)$ has the property $G_\beta(x,t) = G_\beta(t,x)$ which is the assertion (i).

By using the similar discussion as in [1,Proof of Theorem 6], we may obtain

$$G_\beta(x,t) = O(\|t\|^{-n-\beta}),\ \|t\| \to \infty,$$

which shows that (iii) is valid. By (1.5), the assertion (ii) is obvious. (iv) may be obtained by a similar discussion as proof of Theorem 1 in [5].

Lemma 14 *Let $\beta \ge \alpha > \frac{n}{2}$ and $\beta > n$. Then*

(i) $\inf\limits_{h\in S_{\beta,2}} \|f-h\|_{L_2(R^n)} = \sup\left\{ \int_{R^n} f(t) D^\beta g(t)\,dt :\ g \in W_2^\alpha(R^n) \text{ with } Vg = 0\right\}$ holds for any $f \in L_2(R^n)$.

(ii) $E(W_2^\alpha(R^n), S_{\beta,2}, L_2(R^n)) = \sup\left\{ \int_{R^n} D^\alpha f(x) D^{(\beta-\alpha)} g(x)\,dx:\ f \in W_2^\alpha(R^n),\ g \in W_2^\beta(R^n) \text{ with } Vg = 0\right\}$.

Proof First, by the dual theorem in the approximation theory concerning the best approximation (see [13]), we have

$$\inf_{h\in S_{\beta,2}} \|f-h\|_{L_2(R^n)} = \sup\left\{ \int_{R^n} f(t)\varphi(t)\,dt :\ \varphi \in L_p(R^n),\ \varphi \perp S_{\beta,2}\right\}. \tag{3.36}$$

Here the relation $\varphi \perp S_{\beta,2}$ means that

$$\int_{R^n} \varphi(t) h(t)\,dt = 0,\ \text{for all } h \in S_{\beta,2}.$$

For $\varphi \in L_2(R^n)$ with $\varphi \perp S_{\beta,2}$, set $g(x) = \int_{R^n} G_\beta(x,t)\varphi(t)\,dt$. Note that $\gamma(x)$ in $\varphi(R^n)$

implies $\widehat{\gamma}(x)$ in $\varphi(R^n)$. By Plancherel formula, we have

$$\begin{aligned}\int_{R^n} \quad & \|x\|^\beta \widehat{g}(x)\gamma(x)\,\mathrm{d}x = \int_{R^n} g(x)\widehat{\|\cdot\|^\beta\gamma}(x)\,\mathrm{d}x \\ &= \int_{R^n} g(x)D^\beta\widehat{\gamma}(x)\,\mathrm{d}x \\ &= \int_{R^n}\left(\int_{R^n} G_\beta(x,t)\varphi(t)\,\mathrm{d}t\right)D^\beta\widehat{\gamma}(x)\,\mathrm{d}x \\ &= \int_{R^n}\left(\int_{R^n} G_\beta(x,t)D^\beta\widehat{\gamma}(x)\,dx\right)\varphi(t)\,\mathrm{d}t \\ &= \int_{R^n}\left(\widehat{\gamma}(t) - \sum_{\nu\in Z^n}\widehat{\gamma}(\nu)\chi_\beta(t-\nu)\right)\varphi(t)\,\mathrm{d}t \\ &= \int_{R^n}\widehat{\gamma}(t)\varphi(t)\,\mathrm{d}t = \int_{R^n}\widehat{\varphi}(t)\gamma(t)\,\mathrm{d}t\end{aligned}$$

for all $\gamma \in \varphi(R^n)$. Hence, we have $\widehat{\varphi}(t) = \|x\|^\beta\widehat{g}(x)$, which implies $g \in W_2^\beta(R^n)$ and $Vg = 0$. Therefore, by (3.36), we get (i).

Let $f \in W_2^\alpha(R^n)$ and $g \in W_2^\beta(R^n)$ with $Vg = 0$. Then, by the definition of D^β and Plancherel formula[3], we have

$$\begin{aligned}\int_{R^n} \quad & f(x)D^\beta g(x)\mathrm{d}x = \int_{R^n}\widehat{f}(x)\|x\|^\beta\widehat{g}(x)\,\mathrm{d}x \\ &= \int_{R^n}\|x\|^\alpha\widehat{f}(x)\cdot\|x\|^{\beta-\alpha}\widehat{g}(x)\,\mathrm{d}x \\ &= \int_{R^n}(D^\alpha f)(x)D^{\beta-\alpha}g(x)\,\mathrm{d}x.\end{aligned}$$

Thus, by (i) we get (ii).

Proof of Theorem 4 By Theorem 3 and Lemma 14, we have

$$\begin{aligned}& E(W_2^\alpha(R^n), S_{\beta,2}, L_2(R^n)) \\ &\le \sup\{\|D^{(\beta-\alpha)}g\|_{L_2(R^n)} :\ g \in W_2^\beta(R^n), Vg = 0\} \\ &\le \tfrac{1}{\pi^\alpha}.\end{aligned} \tag{3.37}$$

On the other hand, let $f_N(x) = g_N(x) = \mu\left(\frac{x}{N}\right)\sin\pi x_1$ and set

$$F_N(x) = \frac{f_N(x)}{\|D^\alpha f_N\|_{L_2(R^n)}} \quad\text{and}\quad G_N(x) = \frac{g_N(x)}{\|D^\beta g_N\|_{L_2(R^n)}},$$

where $\mu(x)$ is defined in Lemma 10. Then $F_N \in W_2^\alpha(R^n)$ and $G_N \in W_2^\beta(R^n)$with $VG_N = 0$. It is easy to verify that

$$\begin{aligned}\int_{R^n} \quad & D^\alpha F_N(x)D^{(\beta-\alpha)}G_N(x)\,\mathrm{d}x = \frac{\int_{R^n} D^\alpha f_N(x)D^{(\beta-\alpha)}g_N(x)\,dx}{\|D^\alpha f_N\|_{L_2(R^n)}\|D^\beta g_N\|_{L_2(R^n)}} \\ &= \frac{\|D^{\beta/2}f_N\|_{L_2(R^n)}\|D^{\beta/2}g_N\|_{L_2(R^n)}}{\|D^\alpha f_N\|_{L_2(R^n)}\|D^\beta g_N\|_{L_2(R^n)}}.\end{aligned}$$

By (3.26), we have

$$\lim_{N\to\infty} \int_{R^n} (D^\alpha F_N)(x)(D^{(\beta-\alpha)} G_N)(x)\, dx$$
$$= \lim_{N\to\infty} \frac{\|D^{\beta/2} f_N\|_{L_2(R^n)} \|D^{\beta/2} g_N\|_{L_2(R^n)}}{\|D^\alpha f_N\|_{L_2(R^n)} \|D^\beta g_N\|_{L_2(R^n)}} = \frac{1}{\pi^\alpha}. \tag{3.38}$$

Hence, by (3.37), (3.38) and Lemma 14 (ii), we complete the proof of Theorem 4.

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在 $L_2(R^n)$ 尺度下的 Riesz 位势空间上的最优恢复

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摘 要: 考虑了 Riesz 位势空间在给定网格上赋值的多元函数的重构问题，并且得到了 $L_2(R^n)$ 中由 Riesz 位势确定的一些函数类上的精确结果.