Journal of Mathematical Research & Exposition
Vol.20, No.3, 379–380, August 2000

A Note on a Theorem of W.J.LeVeque *

JI Chun-gang, *SHAN Zun*
(Dept. of Math., Nanjing Normal University, Nanjing 210097, China)

Abstract: In this paper we prove that for fixed positive integers α, β, m and $m \geq 2$ the equation $\sum\limits_{j=1}^{n} j^{\alpha} = (\sum\limits_{j=1}^{n} j^{\beta})^{m}$ holds for $n = 2$ only when $m = 2$, $\alpha = 3$ and $\beta = 1$, which is another improvement of LeVeque's result, different from Yu's.

Key words: diophantine equation; identity.

Classification: AMS(1991) 11D72/CLC O156.4

Document code: A **Article ID:** 1000-341X(2000)03-0379-02

In [1] LeVeque proved the following result: If for all positive integers n's there holds an identity of the form

$$\sum_{j=1}^{n} j^{\alpha} = (\sum_{j=1}^{n} j^{\beta})^{m} \tag{1}$$

with fixed positive integers α,β, m and $m \geq 2$, then $m = 2$, $\alpha = 3$ and $\beta = 1$.

In [2] Yu Hongbing generalized LeVeque's result as follows: For fixed positive integers α, β, m and $m \geq 2$, if the equation (1) has infinitely many solutions in positive integers n, then $m = 2$, $\alpha = 3$, and $\beta = 1$.

LeVeque's result rests on his works on Diophantine equations $a^x + 1 = (a^y + 1)^z$ which is more difficult. Yu's proof using the approximate formula: $\sum\limits_{j=1}^{n} j^k \sim \frac{1}{k+1} n^{k+1}$ as $n \to \infty$, the proof is "analytic", not "arithmetical". In this note, we prove the following theorem which is another improvement of LeVeque's result, different from Yu's.

Theorem *For fixed positive integers α, β, m and $m \geq 2$, if the equation (1) holds for $n = 2$, then $m = 2$, $\alpha = 3$, and $\beta = 1$.*

Proof We have

$$1 + 2^{\alpha} = (1 + 2^{\beta})^{m}. \tag{2}$$

Let $m = 2^k t$, where t is odd, then

$$1 + 2^{\alpha} = 1 + 2^{k} \cdot t \cdot 2^{\beta} + \frac{2^k t(2^k t - 1)}{2} 2^{2\beta} + \cdots + \frac{2^k t}{s} \binom{2^k t - 1}{s - 1} 2^{s\beta} + \cdots + 2^{m\beta}. \tag{3}$$

*Received date: 1997-09-15; Revised date: 1999-12-06

Let $v_{(s)}$ denote the order of s at 2, i.e. $2^{v(s)}||s$. If $\beta > 1$, then considering $s \geq 2$, the order of the (s+1)th term at 2 in the right-hand side of (3)

$$\begin{aligned}
&\geq k + s\beta - v_{(s)} \\
&\geq k + \beta + 2(s-1) - v_{(s)} \\
&\geq k + \beta + 2 \cdot 2^{v(s)} - 2 - v_{(s)} \\
&\geq k + \beta + 2(1 + v_{(s)}) - 2 - v_{(s)} \\
&\geq k + \beta + v_{(s)}.
\end{aligned}$$

Furthermore, for $v_{(s)} = 0$, we have $k + s\beta - v_{(s)} = k + s\beta > k + \beta$.

Hence by (3) we get

$$2^\alpha = 2^{k+\beta} \cdot t + 2^{k+\beta+1} \cdot M \quad (M \text{ is a positive integer }).$$

Compare with the orders of both sides of (2) , the last equation is not true. Hence $\beta = 1$. Therefore

$$1 + 2^\alpha = 3^m = 3^{2^k t}$$

implying that

$$2^\alpha = 3^{2^k t} - 1. \tag{4}$$

For $l > 1$, $3^{2^k t} - 1$ has an odd divisor(> 1): $3^{t-1} + 3^{t-2} + \cdots + 1$, but 2^α has not an odd factor(> 1), so $t = 1$.

For $k > 1$, $3^{2^{k-1}} + 1 \equiv 2(\text{mod}4)$. So $3^{2^k} - 1 = (3^{2^{k-1}} + 1)(3^{2^{k-1}} - 1) = 2(\frac{3^{2^{k-1}}+1}{2})(3^{2^{k-1}} - 1)$ has odd divisor(> 1): $\frac{3^{2^{k-1}}+1}{2}$. It is impossible. Hence $k = 1, m = 2$, Consequently $\alpha = 3$. This completes the proof.

Open Problem Prove or disprove the following proposition: If

$$\sum_{j=1}^{n} j^\alpha = \Big(\sum_{j=1}^{n} j^\beta\Big)^m$$

holds for a fixed integer $n > 1$, then $m = 2$, $\alpha = 3$, $\beta = 1$.

References:

[1] LEVEQUE W J. *On the equation* $a^x - b^y = 1$ [J]. Amer. J. Math., 1952, 74: 325–331.

[2] YU Hong-bing. *A note on a theorem of W.J.LeVeque* [J]. J. Math. Res. & Exp., 1994, 14: 537–538.