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The Upper Radical Determined by the Class of all J-Semisimple Subdirectly Irreducible Rings *

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Abstract: F.A.Szasz has put forward the open problem 55 in [1]:

Let **K** be the class of all subdirectly irreducible rings, whose Jacobson radical is (0). Examine the upper radical determined by the class **K**.

In this paper, the problem has been examined.

(1) It has been proved that the upper radical **R** determined by the class **K** is a special radical,which lies between Jacobson radical and Brown-McCoy radical.

(2) It has been given some necessary and sufficient condition of ring A to be an **R**-radical ring.

Key words: subdirectly irreducible ring; special radical; antisimple radical; MHR-ring; Jacobson radical.

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In this paper,any ring is an associative ring and any radical is a radical property in the sense of Amitsur and Kurosh.**J**,**G** and **S** respectively denote Jacobson radical, Brown-McCoy radical and antisimple radical.

Definition 1 *A ring A is called a subdirectly irreducible, if the intersection of all non-zero ideals of A is non-zero. The intersection* **H** *of all non-zero ideals of a subdirectly irreducible ring A is called the heart of A (See[1]).*

Let $\mathbf{K} = \{A|A$ be a subdirectly irreducible ring, $\mathbf{J}\ (A) = 0\}$.

Theorem 1 *The upper radical* **R** *determined by the class* **K** *is a special radical,and thus* **R** *is also a supernilpotent radical.*

Proof First we shall show that A is a subdirectly irreducible ring with heart $\mathbf{H}$, then $\mathbf{J}(A) = 0$ if and only if $\mathbf{J}(\mathbf{H}) = 0$. In fact, if $\mathbf{J}(A) = 0$, since $\mathbf{J}(\mathbf{H}) \subseteq \mathbf{J}(A)$, so $\mathbf{J}(\mathbf{H}) = 0$. Conversely, if $\mathbf{J}(\mathbf{H}) = 0$ but $\mathbf{J}(A) \neq 0$, then $\mathbf{J}(A) \supseteq \mathbf{H}$. Because $\mathbf{J}$ is a hereditary radical,

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so we have $\mathbf{J}(\mathbf{H}) = \mathbf{J}(A) \bigcap \mathbf{H} = \mathbf{H}$. This contradicts the heart $\mathbf{H}$ of A is non-zero. Hence $\mathbf{J}(A) = 0$.

Therefore $\mathbf{K} = \{A|A$ be a subdirectly irreducible ring with a heart $\mathbf{H}, \mathbf{J}(\mathbf{H}) = 0\}$. And then we show that the heart $\mathbf{H}$ of any ring A in $\mathbf{K}$ satisfies $\mathbf{H}^2 = \mathbf{H}$. In fact, because A is a ring in $\mathbf{K}$ and so $\mathbf{J}(\mathbf{H}) = 0$. If $\mathbf{H}^2 \neq \mathbf{H}$, since $\mathbf{H}^2 \subseteq \mathbf{H}$ and from the mininality of $\mathbf{H}$, we obtain $\mathbf{H}^2 = 0$. So $\mathbf{J}(\mathbf{H}) = \mathbf{H}$, a contradiction. Hence $\mathbf{H}^2 = \mathbf{H}$.

Therefore $\mathbf{K} = \{A|A$ be a subdirectly irreducible ring with an idempotent $\mathbf{J}$-semisimple heart $\}$. Thus $\mathbf{K}$ is a special class by theorem 12.3 of [1].So the upper radical determined by the class $\mathbf{K}$ is a special radical, and thus it is also a supernilpotent radical. Theorem 1 is proved.

The upper radical determined by the class $\mathbf{K}$ will be denoted by $\mathbf{R}$.

In the following,we give some characterizations of radical property $\mathbf{R}$.

Let $\mathbf{R}_1$ and $\mathbf{R}_2$ be two radical properties. $\mathbf{R}_1$ and $\mathbf{R}_2$ respectively denote also the class of all $\mathbf{R}_1$-radical rings and the class of all $\mathbf{R}_2$-radical rings. Put $\mathbf{H} = \mathbf{R}_1 \bigcap \mathbf{R}_2$, then the union radical of $\mathbf{R}_1$ and $\mathbf{R}_2$ is $\mathbf{R}_1 + \mathbf{R}_2 = L(\mathbf{H})$ (see[1]).

Theorem 2 *The radical* $\mathbf{R} \geq \mathbf{J} + \mathbf{S}. \mathbf{R} \underset{\neq}{\geq} \mathbf{J}, \mathbf{R} \underset{\neq}{\geq} \mathbf{S}$.

Proof Let $\mathbf{K}_1 = \{A|A$ be a subdirectly irreducuble ring with an idempoent heart $\mathbf{H}\}$, $\mathbf{K}_2 = \{A|J(A) = 0\}$, then $\mathbf{K} = \mathbf{K}_1 \bigcap \mathbf{K}_2$.Since the antisimple radical $\mathbf{S} = U\mathbf{K}_1$, the Jacobson radical $\mathbf{J} = U\mathbf{K}_2$, and $\mathbf{K} \subseteq \mathbf{K}_1$, $\mathbf{K} \subseteq \mathbf{K}_2$, so $\mathbf{R} = U\mathbf{K} \geq U\mathbf{K}_1$, $\mathbf{R} \geq U\mathbf{K}_2$. Consequently, $\mathbf{R} \geq (U\mathbf{K}_1) \bigcup (U\mathbf{K}_2) = \mathbf{J} \bigcup \mathbf{S}$. Therefore $\mathbf{R} \geq L(J \bigcup S) = \mathbf{J} + \mathbf{S}$. Because $\mathbf{J} \nleq \mathbf{S}, \mathbf{S} \nleq \mathbf{J}$ (see[2]), and thus $\mathbf{J} \underset{\neq}{\leq} \mathbf{J} + \mathbf{S}, \mathbf{S} \underset{\neq}{\leq} \mathbf{J} + \mathbf{S}$. Therefore $\mathbf{R} \underset{\neq}{\geq} \mathbf{J}, \mathbf{R} \underset{\neq}{\geq} \mathbf{S}$. This completes the proof of theorem 2.

Theorem 3 $\mathbf{R} \underset{\neq}{\leq} \mathbf{G}$, *and thus* $\mathbf{J} \underset{\neq}{\leq} \mathbf{R} \underset{\neq}{\leq} \mathbf{G}, \mathbf{S} \underset{\neq}{\leq} \mathbf{R} \underset{\neq}{\leq} \mathbf{G}$.

Proof Let $\mathbf{K}_3 = \{A|A$ be a simple ring with a unity element $\}$. Then the Brown-McCoy radical $\mathbf{G} = U\mathbf{K}_3$. Take any $A \in \mathbf{K}_3$, then $\mathbf{G}(A) = 0$. Since $\mathbf{J} < \mathbf{G}$, and so $\mathbf{J}(A) = 0$. Because a simple ring is a subdirectly irreducible ring, thus $A \in \mathbf{K}$. Therefore $\mathbf{K}_3 \subseteq \mathbf{K}$. Consequently $\mathbf{R} = U\mathbf{K} \leq U\mathbf{K}_3 = \mathbf{G}$. By [3], there exists a ring B, which is a $\mathbf{J}$-semisimple simple ring, and also a $\mathbf{G}$-radical ring, and $B \in \mathbf{K}$. Hence $\mathbf{R}(B) = 0$. And thus B is a $\mathbf{R}$-semisimple ring, and also a $\mathbf{G}$-radical ring. Therefore $\mathbf{R} \underset{\neq}{\leq} \mathbf{G}$. Theorem 3 is proved.

This complete the proof that the radical $\mathbf{R}$ determined by the class $\mathbf{K}$ of all subdirectly irreducible rings, whose Jacobson radical is (0), lies between Jacobson radical and Brown-McCoy radical.

Theorem 4 *For an associative ring A, the following conditions are equivalent:*

(1) A is a $\mathbf{R}$*-radical ring.*

(2) Every non-zero homomorphic image A' of A is a subdirect sum of subdirectly irreducible rings with $\mathbf{J}$*-radical heart.*

(3) Every ideal I of A is the intersection of all ideals I_α in the ring A such that each factor ring A/I_α $(I_\alpha \supseteq I)$ is subdirectly irreducible ring with $\mathbf{J}$*-radical heart.*

Proof (1)⇒(2) Let $\mathbf{R}(A) = A$, and A' be a non-zero homomorphic image of A. Then $A' = \sum_S \oplus A_\alpha$, where A_α is a subdirectly irreducible ring. Suppose $\mathbf{H}_\alpha$ is a heart of A_α.

Since $A \sim A' \sim A_\alpha$, $\mathbf{R}(A) = A$, so $\mathbf{R}(A_\alpha) = A_\alpha$. If $\mathbf{J}(\mathbf{H}_\alpha) \neq \mathbf{H}_\alpha$, then $\mathbf{J}(\mathbf{H}_\alpha) = 0$ by proposition 12.2 of [1], because $\mathbf{J}$-radical is a hereditary radical and A_α is a subdirectly irreducible ring with a heart $\mathbf{H}_\alpha$. Therefore $A_\alpha \in \mathbf{K}$, and thus $\mathbf{R} = U\mathbf{K}$. So $\mathbf{R}(A_\alpha) = 0$, in contradiction to $\mathbf{R}(A_\alpha) = A_\alpha$. Consequently $\mathbf{J}(\mathbf{H}_\alpha) = \mathbf{H}_\alpha$, and thus A_α is a subdirectly irreducible ring with a $\mathbf{J}$-radical heart $\mathbf{H}_\alpha$.

(2)$\Rightarrow$(1) Suppose every non-zero homomorphic image A' of A is a subdirect sum of subdirectly irreducible rings with $\mathbf{J}$-radical heart. If $\mathbf{R}(A) \neq A$, then $A/\mathbf{R}(A) \neq 0$. Put $A' = A/\mathbf{R}(A)$. Because $\mathbf{R}(A') = 0, \mathbf{R}$ is a supernilpoent radical, and thus by theorem 11.5 of [1] A' is a subdirect sum of rings in $\mathbf{K}$, $A' = \sum_S \oplus A_\alpha$, where $A_\alpha \in \mathbf{K}$. Therefore every A_α is a subdirectly irreducible ring with a heart $\mathbf{H}_\alpha$ and $\mathbf{J}(\mathbf{H}_\alpha) = 0$. Since $A \sim A' \sim A_\alpha \neq 0$, therefore by assumption, $A_\alpha = \sum_S \oplus A_{\alpha\beta}$, where every $A_{\alpha\beta}$ is a subdirectly irreducible ring with a heart $\mathbf{H}_{\alpha\beta}$, and $\mathbf{J}(\mathbf{H}_{\alpha\beta}) = \mathbf{H}_{\alpha\beta}$. Since A_α is a subdirectly irreducible ring, and so there exists a β_0 such that $A_\alpha \cong A_{\alpha\beta_0}$. And thus $\mathbf{H}_\alpha \cong \mathbf{H}_{\alpha\beta_0}$. But $\mathbf{J}(\mathbf{H}_\alpha) = 0, \mathbf{J}(\mathbf{H}_{\alpha\beta_0}) = \mathbf{H}_{\alpha\beta_0}$, a contradiction. Therefore we must have $\mathbf{R}(A) = A$.

(2)$\Rightarrow$(3) Suppose every non-zero homomorphic image A' of A is a subdirect sum of subdirectly irreducible rings with $\mathbf{J}$-radical heart. Let I be an arbitary ideal of A, then $A \sim A/I, A/I = \sum_S \oplus A_\alpha$, where A_α is a subdirectly irreducible ring with $\mathbf{J}$-radical heart $\mathbf{H}_\alpha$. Hence there exists an ideal $I_\alpha (I_\alpha \supseteq I)$ of A such that $A_\alpha \cong (A/I)/(I_\alpha/I) \cong A/I$, and $\bigcap (I_\alpha/I) = 0$. Therefore A/I_α is a subdirectly irreducible ring with J-radical heart, and $\bigcap I_\alpha = I$.

(3)$\Rightarrow$(2) Suppose every ideal I of A is the intersection of all ideals I_α in the ring A such that each factor ring $A/I_\alpha (I_\alpha \supseteq I)$ is a subdirectly irreducible ring with $\mathbf{J}$-radical heart. If $A \overset{\phi}{\sim} A' \neq 0$, then $I = \ker\phi$ is an ideal of A. By assumption, $I = \bigcap I_\alpha$, where I_α is an ideal of A, A/I_α is a subdirectly irreducible ring with $\mathbf{J}$-radical heart. Hence $\bigcap (I_\alpha/I) = 0$, I_α/I is an ideal of A/I. Therefore $A' \cong A/I = \sum_S \oplus (A/I)/(I_\alpha/I) \cong \sum_S \oplus (A/I_\alpha)$. This completes the proof of theorem 4.

Theorem 4 gives some necessary and sufficient conditions for any associative ring A to be $\mathbf{R}$-radical ring.

Theorem 5 *If the $\mathbf{J}$-radical strongly coincides with the $\mathbf{S}$-radical on the class $\mathbf{M}$ of rings (the class $\mathbf{M}$ is homomorphically closed), then the $\mathbf{R}$-radical also strongly coincides with the $\mathbf{J}$-radical and $\mathbf{S}$-radical on the class $\mathbf{M}$.*

Proof Suppose the $\mathbf{J}$-radical strongly coincides with the $\mathbf{S}$-radical on the class $\mathbf{M}$. Since $\mathbf{R} > \mathbf{J}, \mathbf{R} > \mathbf{S}$, and thus we have only to prove that, for any ring A in $\mathbf{M}$, if $\mathbf{S}(A) = 0$, then $\mathbf{R}(A) = 0$. Let $\mathbf{S}(A) = 0$, then $A = \sum_S \oplus A_\alpha$, where A_α has a heart $\mathbf{H}_\alpha$ and $\mathbf{H}_\alpha^2 = H_\alpha$. Since $\mathbf{S}(A_\alpha) = 0, A_\alpha \in \mathbf{M}$, and so $\mathbf{J}(A_\alpha) = \mathbf{S}(A_\alpha) = 0$. Hence $A_\alpha \in \mathbf{K}$, therefore $\mathbf{R}(A_\alpha) = 0$, and thus $\mathbf{R}(A) = 0$. Theorem 5 is proved.

Definition 2 *A ring A is called a* **MHR**-*ring, if A satisfies the minimum condition on principal right ideals. (see[1])*

Corollary *The radical $\mathbf{R}$ strongly coincides with the radicals $\mathbf{J}$ and $\mathbf{S}$ on the class of all* **MHR**-*rings.The radical $\mathbf{R}$ does not weakly coincide with the radical $\mathbf{G}$ on the class of all*

MHR-*rings*.

Proof Because the radical **J** strongly coincides with the radical **S** on the class of all **MHR**-rings(see[2]),and so the radical **R** also strongly coincides with the radical **J** and **S** on the class of all **MHR**-rings by theorem 5.

Because the radical **J** does not weakly coincides with the radical **G** on the class of all **MHR**-rings(see[4]),while the radical **R** strongly coincides with the radical **J** on the class of all **MHR**-rings. Therefore the radical **R** does not weakly coincide with the radical **G** on the class of all **MHR**-rings.The corollary is proved.

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J-半单亚直既约环类确定的上根

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摘 要: F.A.Szasz 在 [1] 中提出公开问题 55: 设 **K** 是 Jacobson 根为零的全体亚直既约环类，研究类 **K** 确定的上根. 本文对此进行了研究，证明了 Jacobson 根为零的全体亚直既约环类 **K** 确定的上根 **R** 是特殊根，它介于 Jacobson 根与 Brown-McCoy 根之间. 并给出任意结合环 A 为 **R**- 根环的充要条件.