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On Subspaces of Bloch Space *

TAN Hai-ou
(Dept. of Math., Wuyi University, Guangdong 529020, China)

Abstract: In this paper, we give some inclusions on a sort of subspaces of Bloch space, and prove these inclusions are sharp by use of the series with Hadamard gaps. The results containing some known results on Besov spaces and Bloch space.

Key words: Bolch space; series with Hadamard gaps; Besov space.

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Let $D=\{z:|z|<1\}$ be the unit disk of finite complex plane and $A=\{f:f$ is analytic in $D\}$. For $a\in D$, let $\phi_a(z)=\frac{a-z}{1-\bar{a}z}$ is the Möbius mapping of D to itself. For $p>0, q\geq 0$ and $p+q>1$, define

$$E(p,q)=\{f:f\in A \text{ and } \sup_{a\in D}\int_D |f'(z)|^p(1-|z|^2)^{p-2}(1-|\phi_a(z)|^2)^q \mathrm{dm}(z)<\infty\},$$

$$E_0(p,q)=\{f:f\in A \text{ and } \lim_{|a|\to 1}\int_D |f'(z)|^p(1-|z|^2)^{p-2}(1-|\phi_a(z)|^2)^q \mathrm{dm}(z)=0\},$$

where $\mathrm{dm}(z)$ is the Lebesgue measure of D.

As usualy, the letters $\mathcal{B},\mathcal{B}_0,Q_q$and $Q_{q,0}$ denote Bloch, little Bloch,q-Green Dirichlet spaces and little q-Green Dirichlet spaces, respectively, and for $p>1, B_p$ is the classical Besov spaces on D. Then, we have $E(2,q)=Q_q, E_0(2,q)=Q_{q,0}, E(p,0)=B_p(p>1)$ and by [1], $E(p,q)=\mathcal{B}$ and $E_0(p,q)=\mathcal{B}_0$ when $q>1$.

In [2], we have proved the following inclusions:

1. If $p_1<p_2$, then $E(p_1,q)\subset E(p_2,q)$;
2. If $q_1<q_2$, then $E(p,q_1)\subset E(p,q_2)$.

And these inclusions are sharp when $q\leq 1$ (or $q_2\leq 1$).

In the present paper, we will prove another sort of inclusions about $E(p,q)$ and $E_0(p,q)$,and get some results on spaces B_p and Q_q.

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Biography: TAN Hai-ou (1955-), male, born in Hunan province. M.Sc., currently a professor at Wuyi University.

Theorem A *Let $0 \le p_2 \le p_1, 0 \le q_1 < 1$ and $p_1 + q_1 > 1$, then*

(1) $E(p_1, q_1) \subset \bigcap\limits_{\frac{p_1-(1-q_1)p_2}{p_1} \le q_2 \le 1} E(p_2, q_2);$

(2) $E_0(p_1, q_1) \subset \bigcap\limits_{\frac{p_1-(1-q_1)p_2}{p_1} \le q_2 \le 1} E_0(p_2, q_2).$

Proof (1) Let $f \in E(p_1, q_1)$, then

$$\sup_{a\in D} \int_D |f'(z)|^{p_1}(1-|z|^2)^{p_1-2}(1-|\phi_a(z)|^2)^{q_1}\,\mathrm{dm}(z) = C < \infty.$$

Let $\alpha = \frac{p_1q_2-q_1p_2}{p_1-p_2} > 1$, then, by Hölder inequality and [4] (Lemma 4.2.2),

$$\begin{aligned}
&\int_D |f'(z)|^{p_2}(1-|z|^2)^{p_2-2}(1-|\phi_a(z)|^2)^{q_2}\,\mathrm{dm}(z)\\
&\le \left[\int_D |f'(z)|^{p_1}(1-|z|^2)^{p_1-2}(1-|\phi_a(z)|^2)^{q_1}\,\mathrm{dm}(z)\right]^{\frac{p_2}{p_1}} \times\\
&\quad [\int_D (1-|z|^2)^{-2}(1-|\phi_a(z)|^2)^{\alpha}\mathrm{dm}(z)]^{\frac{p_1-p_2}{p_1}}\\
&\le C^{\frac{p_2}{p_1}}[(1-|a|^2)^{\alpha}\int_D \frac{(1-|z|^2)^{\alpha-2}}{|1-\bar{a}z|^{2\alpha}}\mathrm{dm}(z)]^{\frac{p_1-p_2}{p_1}}\\
&\le C^{\frac{p_2}{p_1}} C_1^{\frac{p_1-p_2}{p_1}}.
\end{aligned}$$

where C_1 is an absolute constant. So, we have proved that when $1 \ge q_2 > \frac{p_1-(1-q_1)p_2}{p_1}$, $f \in E(p_2, q_2)$.

(2) Let $f \in E_0(p_1, q_1), \{a_n\} \subset D$ and $|a_n| \to 1$. Then, we need only to prove that: when $\frac{p_1-(1-q_1)p_2}{p_1} < q_2 \le 1$

$$I(n) = \int_D |f'(z)|^{p_2}(1-|z|^2)^{p_2-2}(1-|\phi_{a_n}(z)|^2)^{q_2}\,\mathrm{dm}(z) \to 0. \tag{1}$$

For $a \in D, 0 < \rho < 1$, let $U(a, \rho) = \{z : |\phi_a(z)| < \rho\}$ be the pseudohyperbolic disk with center a and radius ρ, and denote $I(n) = I_1(n) + I_2(n)$, where

$$\begin{aligned}
I_1(n) &= \int_{U(a_n,\rho)} |f'(z)|^{p_2}(1-|z|^2)^{p_2-2}(1-|\phi_{a_n}(z)|^2)^{q_2}\,\mathrm{dm}(z),\\
I_2(n) &= \int_{D\setminus U(a_n,\rho)} |f'(z)|^{p_2}(1-|z|^2)^{p_2-2}(1-|\phi_{a_n}(z)|^2)^{q_2}\,\mathrm{dm}(z).
\end{aligned}$$

Denote $z = \phi_{a_n}(w), f_n(w) = f[\phi_{a_n}(w)]$, we have $(1-|w|^2)|f_n'(w)| = (1-|z|^2)|f'(z)|$.

Because $f \in E_0(p_1, q_1) \subset \mathcal{B}$ (see [2]), so there is a constant $M > 0$, such that $(1-|z|^2)|f'(z)| \le M$, hence, for $w \in D$ and $n = 1, 2, 3, \dots$, we have

$$(1-|w|^2)|f_n'(w)| \le M. \tag{2}$$

Let $\frac{p_1(1-q_2)}{1-q_1} < s < p_2$, using (2) and the equality $|\varphi'_{a_n}(w)| = \frac{1-|\varphi_{a_n}(w)|^2}{1-|w|^2}$, we have

$$
\begin{aligned}
I_2(n) &= \int_{D\setminus U(0,\rho)} |f'_n(w)|^{p_2}(1-|w|^2))^{p_2+q_2-2}\mathrm{dm}(w)\\
&\le M^{p_2-s}\int_{D\setminus U(0,\rho)} |f'_n(w)|^{s}(1-|w|^2)^{p_2+s-2}\mathrm{dm}(w)\\
&\le M^{p_2-s}[\int_D |f'(z)|^{p_1}(1-|z|^2)^{p_1-2}(1-|\phi_{a_n}(z)|^2)^{q_1}\mathrm{dm}(z)]^{\frac{s}{p_1}}\times\\
&[2\pi\int_\rho^1 (1-r^2)^{\frac{q_2p_1-q_1s}{p_1-s}-2} r\mathrm{d}r]^{\frac{p_1-s}{p_1}}\\
&\le C_2(1-\rho^2)^{\frac{q_2p_1-q_1s-p_1+s}{p_1}},
\end{aligned}
$$

where C_2 is a constant independent of ρ and n. Hence, for any $\varepsilon > 0$, there is $\rho_0, 0 < \rho_0 < 1$, such that, for $\rho_0 \le \rho < 1$, we have $I_2(n) < \frac{\varepsilon}{2}, (n = 1, 2, 3, \cdots)$.

Fixed a $\rho \in (\rho_0, 1)$, and denote $C_3 = \int_{U(0,\rho)} \frac{1}{(1-|w|^2)^{2-q_2}}\mathrm{dm}(w)$, because $f \in E_0(p_1, q_1) \subset \mathcal{B}_0$ ([2]), so, there is a $r_0 : 0 < r_0 < 1$,

$$
(1-|z|^2)^{p_2}|f'(z)|^{p_2} \le \frac{\varepsilon}{2C_3} \tag{3}
$$

for $r_0 < |z| < 1$, and there is positive integer n_0, such that, when $n > n_0$, if $z \in U(a_n, \rho)$, then $|z| > r_0$. Now, let $n > n_0$ and using (3), we have

$$
\begin{aligned}
I_1(n) &= \int_{U(a_n,\rho)} |f'(z)|^{p_2}(1-|z|^2)^{p_2-2}(1-|\phi_{a_n}(z)|^2)^{q_2}\mathrm{dm}(z)\\
&\le \frac{\varepsilon}{2C_3}\int_{U(a_n,\rho)} \frac{(1-|\phi_{a_n}(z)|^2)^{q_2}}{(1-|z|^2)^2}\mathrm{dm}(z) = \frac{\varepsilon}{2}.
\end{aligned}
$$

This proved $\lim\limits_{n\to\infty} I(n) = 0$ and $f \in E_0(p_2, q_2)$.

In [2], we have proved the following results:

Let $f(z) = \sum\limits_{k=0}^{\infty} a_k z^{n_k} \in A$ and satisfying $\frac{n_{k+1}}{n_k} \ge \lambda > 1 (k = 1, 2, 3, \cdots)$. Then, the following statements are equivalent:

(a) $f \in E(p, q)$.

(b) $f \in E_0(p, q)$.

(c) $\sum\limits_{n=0}^{\infty} 2^{n(1-q)}(\sum\limits_{n_k\in I_n} |a_k|^p) < \infty$, where $I_n = \{k : 2^n \le k < 2^{n+1}, k \in \mathbf{N}\}$.

Using these results, we can show that:

1) In Theorem A, the inclusions are sharp. In fact, let $f_1(z) = \sum\limits_{n=0}^{\infty} a_n z^{2^n}, a_n = \frac{1}{2^{\frac{n(1-q_1)}{p_1}}}$. Then $f_1 \in E_0(p_2, q_2)\setminus E(p_1, q_1)$.

2) In Theorem A, the low bound $\frac{p_1-(1-q_1)p_2}{p_1}$ is the best. In fact, for $q_2 \leq \frac{p_1-(1-q_1)p_2}{p_1}$, let $f_2(z) = \sum_{n=0}^{\infty} a_n z^{2^n}, a_n = \frac{1}{(n+1)^{\frac{1}{p_2}} 2^{\frac{n(1-q_1)}{p_1}}}$. Then $f_2 \in E_0(p_1, q_1) \backslash E(p_2, q_2)$.

In Theorem A, let $p_2 = 2, q_1 = 0$ and replace q_2 by q, p_1 by p, we have

Corollary 1 *Suppose $p > 2$, then $B_p \subset \bigcap_{\frac{p-2}{p} < q \leq 1} Q_{q,0}$.*

Because $B_p \subset B_q$ when $1 < p < q$, we get

Corollary 2 *If $1 < p \leq 2$, then $B_p \subset \bigcap_{0<q\leq 1} Q_{q,0}$.*

Corollary 3 *Suppose $0 < p_2 < p_1$, then*
1) $E(p_1, \frac{1}{p_1}) \subset E(p_2, \frac{1}{p_2})$; $E_0(p_1, \frac{1}{p_1}) \subset E_0(p_2, \frac{1}{p_2})$ for $\frac{1}{p_1} + \frac{1}{p_2} > 1$.
2) above inclusions don't valid if $\frac{1}{p_1} + \frac{1}{p_2} \leq 1$.

Proof 1) If $0 < p_2 < 1$, then $E(p_2, \frac{1}{p_2}) = \mathcal{B}$ and $E_0(p_2, \frac{1}{p_2}) = \mathcal{B}_0$. Hence the corollary is true by [2].

If $p_2 \geq 1$ we choose $q_1 = \frac{1}{p_1}, q_2 = \frac{1}{p_2}$ in Theorem A know that the results is true.

2) Now, suppose $\frac{1}{p_1} + \frac{1}{p_2} \leq 1$, let $f_3(z) = \sum_{n=0}^{\infty} a_n z^{2^n} \in A$, where $a_n = \frac{1}{(n+1)^{\frac{1}{p_2}} 2^{\frac{n(1-\frac{1}{p_1})}{p_1}}}$, one can prove that $f_3 \in E_0(p_1, \frac{1}{p_1}) \backslash E(p_2, \frac{1}{p_2})$.

The author imagine that the inclusions $E(p_2, \frac{1}{p_2}) \subset E(p_1, \frac{1}{p_1})$, $E_0(p_2, \frac{1}{p_2}) \subset E_0(p_1, \frac{1}{p_1})$ should be vaild for $\frac{1}{p_1} + \frac{1}{p_2} \leq 1, (p_2 < p_1)$.

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关于 Bloch 空间的子空间

谭 海 鸥

(五邑大学，广东江门 529020)

摘 要: 本文给出了 Bloch 空间一类子空间的包含关系，并利用 Hadamard 缺项级数证明了这些关系是最好的. 所获得的结果包含了 Besov 空间和 Bloch 空间的一些已知结论.