Journal of Mathematical Research & Exposition
Vol.20, No.4, 515–522, November, 2000

Doob's Stopping Theorems for Set-Valued (Super, Sub) Martingales with Continuous Time *

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Abstract: In this paper the regularity of set-valued martingales in the sense of J_L is given first. Then we show some kinds of Doob's stopping theorems for set-valued (super, sub) martingales with continuous time.

Key words: set-valued (super, sub) martingale; Doob's stopping theorem; set-valued conditional expectation; regularity.

Classification: AMS(1991) 60D05,60G40,60G42/CLC O211.61

Document code: A **Article ID:** 1000-341X(2000)04-0515-08

1. Introduction and Preliminaries

Doob's stopping theorems and optional sampling theorems for set-valued martingales were first studied by [1]. [10] proved an optional sampling theorem for set-valued martingales by virtue of [8], extending an earlier result of [1]. [13] established Doob's stopping theorems for set-valued (super, sub) martingales in $\mathcal{L}^1_{fc}(X)$. [12] established all kinds of (super, sub) martingales, extending and improving results of [10] and [13]. The purpose of the present paper goes on with the study of stopping theorems for multivalued martingales with continuous time.

Let $(X, \|\cdot\|)$ be a separable Banach space with the dual X^*, and 2^X the set of all subsets of X. Put

$$P_f(X) = \{A \in 2^X \backslash \emptyset : A \text{ is closed}\},$$
$$P_{(b)fc}(X) = \{A \in P_f(X) : A \text{ is (bounded) convex}\},$$
$$P_{wkc}(X) = \{A \in P_{fc}(X) : A \text{ is weakly compact}\}.$$

*Received date: 1998-09-14
Foundation item: Supported by the National Natural Science Foundation of China (19971072)
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For $A \in 2^X$, we denote by cl$A$ and $\overline{\text{co}}A$ the closure and the closed convex hull of A respectively, and define $\| A \| = \sup\{\| x \| : x \in A\}$,

$$d(x, A) = \inf\{\|x - y\| : y \in A\}, \qquad d(x, \emptyset) = \infty, \qquad x \in X,$$
$$s(x^*, A) = \sup\{< x^*, y >: y \in A\}, \qquad s(x^*, \emptyset) = -\infty, \qquad x^* \in X^*.$$

$s(x^*, A)$ and $d(x, A)$ are called the support function and the distance function of A respectively. For $A, B \in 2^X$, put $h^+(A, B) = \sup\{d(a, B) : a \in A\}$, $h^-(A, B) = \sup\{d(b, A) : b \in B\}$. The Hausdorff metric h on $P_f(X)$ is defined by

$$h(A, B) = \max\{h^+(A, B), h^-(A, B)\}, \qquad A, B \in P_f(X).$$

Then $(P_f(X), h)$ is a complete metric spaces.

Let $(\Omega, \mathcal{F})$ be a measurable space. The collection of all $\mathcal{F}$-measurable random sets is denoted by $M[\mathcal{F}; P_f(X)]$. Similarly, the collection of all $P_{fc}(X)$-valued ($P_{wkc}(X)$-valued) random sets is denoted by $M[\mathcal{F}; P_{fc}(X)]$($M[\mathcal{F}; P_{wkc}(X)]$).

Suppose that $(\Omega, \mathcal{F}, P)$ is a complete probability space and $\mathcal{G}$ is a sub-σ-field of $\mathcal{F}$. $L^1(\Omega, \mathcal{G}, P; X)$ is the set of all $\mathcal{G}$-measurable Bochner integrable random elements. We also simplify $L^1(\Omega, \mathcal{F}, P; X)$ as $L^1(\Omega; X)$. For $F \in M[\mathcal{F}; P_f(X)]$, set

$$S_F^1(\mathcal{G}) = \{f \in L^1(\Omega, \mathcal{G}, P; X) : f(\omega) \in F(\omega) \quad \text{a.s.}\},$$

and $S_F^1(\mathcal{F})$ is often written by S_F^1. For random set F, if $S_F^1 \neq \emptyset$, then we call F integrable. It is easy to show that $S_F^1 \neq \emptyset$ if and only if $E(0, F) < \infty$. Put

$$\mathcal{L}_f^1(X) = \left\{F \in M[\mathcal{F}; P_f(X)] : \quad \int_\Omega \| F \| \, dP < \infty\right\},$$
$$\mathcal{L}_{fc}^1(X) = \{F \in \mathcal{L}_f^1(X) : \quad F(\omega) \in P_{fc}(X) \text{ a.s.}\},$$
$$L_{wkc}^1(X) = \{F \in \mathcal{L}_f^1(X) : \quad F(\omega) \in P_{wkc}(X) \text{ a.s.}\}.$$

Let $F \in M[\mathcal{F}; P_f(X)]$, if $S_F^1 \neq \emptyset$, then the integral of F is defined by

$$\int_\Omega F dp = \{\int_\Omega f dp : \ f \in S_F^1\},$$

where $\int_\Omega f dp$ is the Bochner integral. This definition was a natural generalization of point-valued function. For $A \in \mathcal{F}$, $\int_A F dp$ is the integral of the restriction of F on A. The expectation of F is defined by $EF = \text{cl} \int_\Omega F dp = \text{cl}\{\int_\Omega f dp : \ f \in S_F^1\}$.

Put

$$\mathcal{L}_f^{d1}(X) = \{F \in M[\mathcal{F}; P_f(X)] : \ Ed(0, F) < \infty\},$$
$$\mathcal{L}_{fc}^{d1}(X) = \{F \in M[\mathcal{F}; P_{fc}(X)] : Ed(0, F) < \infty\}.$$

Then $\mathcal{L}_f^{d1}(X)$ is the collection of all integrable random sets . The conditional expectation of $F \in \mathcal{L}_f^{d1}(X)$ with respect to $\mathcal{A}$, $E(F|\mathcal{A})$, is the unique (up to a P-null set) $\mathcal{A}$-measurable random set in $\mathcal{L}_f^{d1}$ such that $S^1_{E(F|\mathcal{A})}(\mathcal{A}) = \text{cl}\{E(f|\mathcal{A}) : \ f \in S^1_F\}$, the closure taken in $L^1(\Omega; X)$ (see [7] Theorem 5.17). Note that if $\mathcal{A}$ is trivial, i.e., $\mathcal{A} = \{\emptyset, \Omega\}$, then

$$E(F|\mathcal{A}) = EF = cl \int_\Omega F dp.$$

The conditional expectation of a random closed set behaves much like the traditional single-valued conditional expectation, for more details we may refer to [7]. Put

$$P_{Rwkc}(X) = \{A \in P_{fc}(X) : A \hat{\cap} B(0,r) \in P_{wkc}(X), r > 0\},$$

where $B(0,r)$ is the closed ball of radius r, centered at 0.

The rest of this paper is organized as follows: In section 2 the regularity of multivalued martingales is proved in the sense of J_L. Doob's stopping theorems are discussed in section 3.

2. Regularity of Set-valued martingales in the sense of J_L.

In this section, let $(\Omega, \mathcal{F}, P)$ be a complete probability space, a filtration with continuous time $(\mathcal{F}_t, t \geq 0)$ is given. Also $(\mathcal{F}_t, t \geq 0)$ satisfies the usual conditions, $\mathcal{F}_\infty = \vee_{t\geq 0}\mathcal{F}_t$. By $\overline{T}$ (resp. T_f, T) we will denote the set of all $(\mathcal{F}_t)$-(resp. finite, bounded) stopping times. Before we set up a theorem concerned with the regularity of continuous parameter set-valued martingales, we introduce some notions of convergence of a family of sets in $P_f(X)$. Let $(A, A_r, r \geq 0) \subset P_f(X)$, put

$$\begin{aligned}
w - \underline{\lim}_{r\to t} A_r &= \{x \in X : \exists x_r \in A_r, r \geq 0, \text{s.t. } x_r \xrightarrow{w} x, r \to t\},\\
w - \overline{\lim}_{r\to t} A_r &= \{x \in X : \exists x_{r_n} \in A_{r_n}, n \geq 1, r_n \to t, n \to \infty, \text{s.t. } x_{r_n} \xrightarrow{w} x, n \to \infty\},\\
s - \underline{\lim}_{r\to t} A_r &= \{x \in X : \exists x_r \in A_r, \text{s.t. } x_r \xrightarrow{s} x, r \to t\},\\
s - \overline{\lim}_{r\to t} A_r &= \{x \in X : \exists x_{r_n} \in A_{r_n}, n \geq 1, r_n \to t, n \to \infty, \text{s.t. } x_{r_n} \xrightarrow{s} x, n \to \infty\}.
\end{aligned}$$

Here s-denotes the strong topology on X and w-the weak topology. Note that we always have

$$s - \underline{\lim}_{r\to t} A_r \subset s - \overline{\lim}_{r\to t} A_r \subset w - \overline{\lim}_{r\to t} A_r, \quad s - \underline{\lim}_{r\to t} A_r \subset w - \underline{\lim}_{r\to t} A_r \subset w - \underline{\lim}_{r\to t} A_r.$$

Definition 2.1 *(1) If* $\lim_{r\to t} h(A_r, A) = 0$, *then we say* (A_r) *convergent to* A *at* t *in the Hausdorff metric and denote by* $(h)\lim_{r\to t} A_r = A$;

(2) If $s - \underline{\lim}_{r\to t} A_r = w - \overline{\lim}_{r\to t} A_r = A$, *then we call* (A_r) *convergent to* A *at* t *in the Kuratowski Mosco sense and denote by* $(K - M)\lim_{r\to t} A_r = A$;

(3) If $s - \underline{\lim}_{r\to t} A_r = s - \overline{\lim}_{r\to t} A_r = A$, *then we call* (A_r) *convergent to* A *at* t *in the Kuratowski sense and denote by* $(K)\lim_{r\to t} A_r = A$;

(4) *If* $w - \underline{\lim}_{r\to t} A_r = w - \overline{\lim}_{r\to t} A_r = A$, *then we say* (A_r) *convergent to* A *at* t *in the* τ_w *sense and denote by* $(\tau_w)\lim_{r\to t} A_r = A$;

(5) *If for each* $x^* \in X^*, \lim_{r\to t} s(x^*, A_r) = s(x^*, A)$, *then we say* (A_r) *weakly convergent to* A *at* t *and denote by* $(w)\lim_{r\to t} A_r = A$;

(6) *If for each* $x \in X, \lim_{r\to t} d(x, A_r) = d(x, A)$, *then we call* (A_r) *convergent to* A *at* t *in the Wijsman sense and denote by* $(Wijs)\lim_{r\to t} A_r = A$;

(7) *If* $(w)\lim_{r\to t} A_r = A$, $(Wijs)\lim_{r\to t} A_r = A$, *then we say* (A_r) *convergent to* A *at* t *in the* J_L*-sense and denote by* $(J_L)\lim_{r\to t} A_r = A$.

Obviously, the notations of $K - M$ and weak convergence of sets are in general disjoint and are both implied by convergence in the Hausdorff metric. Also

$$(K - M)\lim_{r\to t} A_r = A \to (k)\lim_{r\to t} A_r = A; \ (K - M)\lim_{r\to t} A_r = A \to (\tau_w)\lim_{r\to t} A_r = A.$$

Lemma 2.2 *Let* $(A_r, r \geq 0) \subset P_f(X)$. *Then*

$$s - \underline{\lim}_{r\to t} A_r = \{x \in X : \lim_{r\to t} d(x, A_r) = 0\}.$$

Proof Take $x \in s - \underline{\lim}_{r\to t} A_r$. There exist $x_r \in A_r, r \geq 0$ such that $\|x_r - x\| \to 0, r \to t$. Thus $d(x, A_r) \leq \|x_r - x\| \to 0, r \to t$. Hence $\lim_{r\to t} d(x, A_r) = 0$. Conversely, take $x \in \{x \in X : \lim_{r\to t} d(x, A_r) = 0\}$. For $r \geq 0$, there exists $x_r \in A_r$ shch that $\|x - x_r\| \leq d(x, A_r) + |r - t|$. Letting $r \to t$ gives $\|x - x_r\| \to 0$. Therefore, $x \in s - \underline{\lim}_{r\to t} A_r$. The desired conclusion is proved.

The proofs of following two lemmas are completely similar to those of analogous theorems in [2] [13], and are omitted.

Lemma 2.3 *Let* $\{A, A_r, r \geq 0\} \subset P_f(X), A_r \subset G \in P_{wk}(X), r \geq 0$. *If* $(K - M)\lim_{r\to t} A_r = A$, *then* $(Wijs)\lim_{r\to t} A_r = A$.

Lemma 2.4 *Let* $\{A, A_r, r \geq 0\} \subset P_f(X), A_r \subset G \in P_{wk}(X), r \geq 0, A \subset G$. *If* $(\tau_w)\lim_{r\to t} A_r = A$, *then* $(w)\lim_{r\to t} A_r = A$.

Proposition 2.5 *Let* $(A, A_r, r \geq 0) \subset P_{fc}(X), A_r \subset G \in P_{wk}(X), r \geq 0$. *Then the following statements are equivalent:*

(1) $(J_L)\lim_{r\to t} A_r = A$;

(2) $(K - M)\lim_{r\to t} A_r = A$.

Proof (1) → (2): For $x \in A$, (1) imlpies $\lim_{r\to t} d(x, A_r) = d(x, A) = 0$. Applying Lemma 2.2 we have $x \in s - \underline{\lim}_{r\to t} A_r$. This means $A \subset s - \underline{\lim}_{r\to t} A_r$. Furthermore, by Proposition 1.2 of [11] and (1) we deduce that $w - \overline{\lim}_{r\to t} A_r \subset A$. Hence $A \subset s - \underline{\lim}_{r\to t} A_r \subset w - \overline{\lim}_{r\to t} A_r \subset A$. This yields (2) holds.

(2) → (1): (2) and Lemma 2.3 imply $(Wijs)\lim_{r\to t} A_r = A$. On the other hand, since $(K-M)\lim_{r\to t} A_r = A$, we obtain $(\tau_w)\lim_{r\to t} A_r = A$. Using Lemma 2.4 we get $(w)\lim_{r\to t} A_r = A$. Hence (2) holds. □

Immediately from Proposition 2.5 and Theorem 5.3 of [3], we can easily prove the following theorem concerned with the regularity of continuous parameter multivalued martingales in the sense of J_L.

Theorem 2.6 *Suppose that $\{F_t, \mathcal{F}_t, t \in R_+\}$ is a set-valued martingale in $\mathcal{L}^1_{wkc}(X)$, and there exists $H \in \mathcal{L}^1_{wkc}(X)$ such that $F_t(\omega) \subset H(\omega), \omega \in \Omega, t \geq 0$, S is a countable dense subset of R_+ containing 0. Then there exists an adapted process $\{\tilde{F}_t, \mathcal{F}_t, t \in R_+\}$ in $\mathcal{L}^1_{wkc}(X)$ such that*

(1) $\{\tilde{F}_t, t \in R_+\}$ is J_L-right continuous and for almost all ω

$$\tilde{F}_t(\omega) = (J_L)\lim_{r\downarrow t, r\in S} F_r(\omega), \quad t \geq 0;$$

(2) For almost all ω, $\tilde{F}_{t-}(\omega) = (J_L)\lim_{r\uparrow t, r\in R_+} \tilde{F}_r(\omega)$ exists for each $t > 0$ and

$$\tilde{F}_{t-}(\omega) = (J_L)\lim_{r\uparrow t, r\in S} F_r(\omega);$$

(3) For each $t \geq 0$, $\tilde{F}_t = F_t$ a.s.;

(4) $\{\tilde{F}_t, \mathcal{F}_t, t \in R_+\}$ is a multivalued martingale.

3. Doob's stopping theorem for set-valued (super, sub)martingales with continuous time

Definition 3.1 *A $P_{fc}(X)$-valued (super, sub) martingale $(F_t, t \in R_+)$ is called right-closable, if there exists an integrable $P_{fc}(X)$-valued random set $F_\infty \in \mathcal{F}_\infty$ such that for each $t \in R_+$, $E(F_\infty|\mathcal{F}_t) = (\subset, \supset)F_t$ a.s.. In this case $(F_t, t \in \overline{R}_+)$ is called a right-closed set-valued (super, sub) martingale, and F_∞ is the right-closing element of $(F_t, t \in R_+)$.*

When $(F_t, t \in \overline{R}_+)$ is a martingale in $L^1_{fc}(X)$; or when X^* is separable and $(F_t, t \in \overline{R}_+)$ is a martingale in $\mathcal{L}^1_{fc}(X)$; or when $(F_t, t \in \overline{R}_+)$ is a martingale in $\mathcal{L}^1_{wkc}(X)$, immediately from [7], [6] we see that for a right-closed set-valued martigale the right-closing element is uniquely determined. By virtue of properties of support function one can easily prove the following theorem, and so is omitted.

Theorem 3.2 *Suppose that $(F_t, t \in \overline{R}_+)$ is a $P_{fc}(X)$-valued w-right continuous martingale. If there exists a $H \in \mathcal{L}^1_{wkc}(X)$ such that $F_t(\omega) \subset H(\omega), \omega \in \Omega, t \in \overline{R}_+$, then for $S, \tau \in \overline{T}, S \leq \tau$ we have*

$$E(F_\tau|\mathcal{F}_S) = F_S \quad a.s.. \tag{3.2.1}$$

The following theorem is a strengthened form of above Theorem 3.2.

Theorem 3.3 *Suppose that $(F_t, t \in \overline{R}_+)$ is a $P_{fc}(X)$-valued w-right continuous martingale. If there exists a $H \in \mathcal{L}^1_{wkc}(X)$ such that $F_t(\omega) \subset H(\omega), \omega \in \Omega, t \in \overline{R}_+$, then for*

$S, \tau \in \overline{T}$, *we have* $E(F_\tau|\mathcal{F}_S) = F_{S\wedge\tau}$ *a.s.*.

Proof By Theorem 3.2 we know $F_\tau \in \mathcal{L}^1_{wkc}(X)$. Since $F_\tau = F_\tau I_{[\tau\le S]} \dot{+} F_{\tau\vee S} I_{[\tau>S]}$, using Theorem 3.2 we obtain

$$\begin{aligned} E(F_\tau|\mathcal{F}_S) &= E(F_\tau I_{[\tau\le S]} \dot{+} F_{\tau\vee S} I_{[\tau>S]}|\mathcal{F}_S) \\ &= E(F_\tau I_{[\tau\le S]}|\mathcal{F}_S) \dot{+} E(F_{\tau\vee S} I_{[\tau>S]}|\mathcal{F}_S) \\ &= F_\tau I_{[\tau\le S]} \dot{+} E(F_{\tau\vee S}|\mathcal{F}_S) I_{[\tau>S]} \\ &= F_\tau I_{[\tau\le S]} \dot{+} F_S I_{[\tau>S]} = F_{S\wedge\tau} \quad \text{a.s..} \end{aligned}$$

Corollary 3.4 *Let* $F \in \mathcal{L}^1_{wkc}(X)$. *Then for* $S, \tau \in \overline{T}$ *we have*

$$E(E(F|\mathcal{F}_S)|\mathcal{F}_\tau) = E(E(F|\mathcal{F}_\tau)|\mathcal{F}_S) = E(F|\mathcal{F}_{S\wedge\tau}) \quad a.s.. \tag{3.4.1}$$

Proof Using Theorem 2.6 and Corollary 2.60[4] we can prove it easily. □

Next we discuss Doob's stopping Theorem for unbounded set-valued supermartingale with continuous time. To this end,we give the following two definitions.

Definition 3.5 *Let* $(A_t, t \ge 0) \subset P_f(X)$. $\{A_t, t \ge 0\}$ *is called RW continuous at* t_0, *if for each* $r > \sup_{t\ge 0} d(0, A_t)$ *we have*

$$\lim_{t\to t_0} s(x^*, A_t \cap B(0,r)) = s(x^*, A_{t_0} \cap B(0,r)), \qquad x^* \in X^*.$$

If $\{A_t, t \ge 0\}$ *is RW-continuous for all* $t \in R_+$, *then we say* $\{A_t, t \ge 0\}$ *RW-continuous. Similarly, we can define RW right continuous.*

Obviously, when $\{A_t, t \ge 0\}$ is bounded, (namely, there exists $r > 0$ such that $\sup_{t\ge 0} \|A_t\| < r$) $\{A_t, t \ge 0\}RW$-(right) continuous $\to$ $\{A_t, t \ge 0\}w$-(right) continuous.

Definition 3.6 *If* $\{A_t, t \ge 0\} \subset P_f(X)$ *is both Wijs-right continuous and RW-right continuous, then we call* $\{A_t, t \ge 0\} J^*_L$*-right continuous. Let* $\{F_t, t \ge 0\}$ *be a set-valued process, if for each* $\omega \in \Omega, \{F_t(\omega), t \ge 0\}$ *is* J^*_L*-right continuous, then we say* $\{F_t, t \ge 0\}$ J^*_L*-right continuous.*

Lemma 3.7 *Let* $(F_t, \mathcal{F}_t; t \in \overline{R}_+)$ *be a* $P_{fc}(X)$*-valued* J_L*-right continuous supermartingale. Then for* $S \in \overline{T}$, *we have* $F_S \in \mathcal{L}^{d1}_{fc}(X)$. *In addition, for* $S, \tau \in \overline{T}, S \le \tau$, *if one of the following conditions is satisfied:*

(i) $\{F_t, t \in \overline{R}_+\} \subset \mathcal{L}^1_{fc}(X), E\|F_\tau\| < \infty, E\|F_S\| < \infty, X^*$ *is separrable;*

(ii) $\{F_t, t \in \overline{R}_+\} \subset \mathcal{L}^1_{wkc}(X), E\|F_\tau\| < \infty, E\|F_S\| < \infty,$

then we have $E(F_\tau \mathcal{F}_S) \subset F_S$ *a.s.*.

Proof It is completely similar to those of Proposition 3.15 in [12] and Theorem 3.2, and so is omitted. □

Now we turn our attention to Doob's stopping theorem for set-valued right-closed supermartingale whose values may be unbounded.

Theorem 3.8 *Assume that* $(F_t, \mathcal{F}_t, t \in \overline{R}_+)$ *is a* J^*_L*-right continuous* $P_{fc}(X)$*-valued* *right-closed supermartingale. Then for* $S \in \overline{T}$, *we have* $F_S \in \mathcal{L}^{d1}_{fc}(X)$. *In addition, if* X^* *is separable, or* $F_t(\omega) \in P_{Rwkc}(X), \omega \in \Omega, t \in \overline{R}_+$, *then for* $S, \tau \in \overline{T}, S \leq \tau$ *we have*

$$E(F_\tau|\mathcal{F}_S) \subset F_S \quad a.s.. \tag{3.8.1}$$

Proof Assume $(F_t, \mathcal{F}_t, t \in \overline{R}_+)$ is J^*_L-right continuous, by Definition 3.6 we deduce $\{d(x, F_t), t \in \overline{R}_+\}$ is a right continuous adapted process for each $x \in X$. Thus $\{d(x, F_t), t \in \overline{R}_+\}$ is a progressive process by [4] Theorem 3.11. Using [4] Theorem 3.12 we conclude $d(x, F_S) \in \mathcal{F}_S$ for each $x \in X$. Thus F_S is measurable w.r.t. $\mathcal{F}_S$. Because $(F_t, \mathcal{F}_t, t \in \overline{R}_+)$ is a right-closed supermartingale, using [5] we know $\{d(0, F_t), \mathcal{F}_t, t \in \overline{R}_+\}$ is a non-negative right -closed submartingale. In particular $E(d(0, F_\infty|lF_t) \geq d(0, F_t)$ a.s., $t \geq 0$. Set

$$\xi_t = E(d(0, F_\infty)|\mathcal{F}_t) + 1, t \in \overline{R}_+.$$

Since the filtration $(\mathcal{F}_t, t \geq 0)$ satisfies the usual conditions, $\{\xi_t, t \in \overline{R}_+\}$ has a right-continuous adapted modification. For simplicity, we still denote it by $(\xi_t, t \in \overline{R}_+)$. Put $G^k_t = B(0, k\xi_t), t \in \overline{R}_+, k \geq 1$. Evidently, the above argument shows $F_t \cap G^k_t \neq \emptyset, k \geq 1, t \in \overline{R}_+$. For $s < t, k \geq 1$, using [5] Lemma 4.2 we obtain

$$\begin{aligned} E(F_t \cap G^k_t|\mathcal{F}_s) &\subset E(F_t|\mathcal{F}_s) \cap E(G^k_t|\mathcal{F}_s) \\ &\subset F_s \cap E(B(0, k\xi_t)|\mathcal{F}_s) \\ &= F_s \cap B(0, E(k\xi_t|\mathcal{F}_s)) \\ &= F_s \cap B(0, k\xi_s) = F_s \cap G^k_s \quad \text{a.s..} \end{aligned}$$

Similarly, $E(F_\infty \cap G^k_\infty|\mathcal{F}_s) \subset F_s \cap G^k_s$. Hence $(F_t \cap G^k_t, t \in \overline{R}_+)$ is a $P_{fc}(X)$-valued J_L-right continuous supermartingale. On the other hand, since

$$\begin{aligned} &\|F_\tau \cap G^k_\tau\| \leq \|G^k_\tau\| \leq k\xi_\tau = k(E[d(0, F_\infty)|\mathcal{F}_\tau] + 1), \\ &\|F_S \cap G^k_S\| \leq k(E[d(0, F_\infty)|\mathcal{F}_S] + 1), \quad k \geq 1, \end{aligned}$$

we get $E\|F_\tau \cap G^k_\tau\| < \infty$, $E\|F_S \cap G^k_S\| < \infty$. Therefore, by the monotone convergence theorem of conditional expecation [6] and Lemma 3.7 we get

$$\begin{aligned} E(F_\tau|\mathcal{F}_S) &= E[\bigcup_{k=1}^{\infty} (F_\tau \cap G^k_\tau)|\mathcal{F}_S] \\ &= \text{cl}[\bigcup_{k=1}^{\infty} E(F_\tau \cap G^k_\tau|\mathcal{F}_S)] \\ &\subset \text{cl}[\bigcup_{k=1}^{\infty} (F_S \cap G^k_S)] = F_S. \end{aligned}$$

Thus (3.8.1) is established. □

Corollary 3.9 *Assume that* $(F_t, \mathcal{F}_t, t \in \overline{R}_+)$ *is a* J^*_L*-right continuous* $P_{fc}(X)$*-valued right-closed supermartingale. If* X^* *is separable or* $F_t(\omega) \in P_{Rwkc}(X), \omega \in \Omega, t \in \overline{R}_+$, *then for* $S, \tau \in \overline{T}$, *we have*

$$E(F_\tau|\mathcal{F}_S) \subset F_{S\wedge\tau} \quad a.s..$$

Finally, we give the Doob's stopping theorem for right-closed set-valued submartingale to end this section. The proof of it is totally similar to that of Theorem 3.2, and is omitted.

Theorem 3.10 *Let $(F_t, t \in \overline{R}_+)$ be a w-right continuous right-closed submartingale, and there exists $H \in \mathcal{L}^1_{wkc}(X)$ such that $F_t(\omega) \subset H(\omega), \omega \in \Omega, t \in \overline{R}_+$. Then for $S, \tau \in \overline{T}, S \leq \tau$ we have*

$$E(F_\tau | \mathcal{F}_S) \supset F_S \quad a.s..$$

References:

[1] ALO R, DE KORVIN A and ROBERTS C. *The optional sampling theorem for convex set-valued martingales* [J]. J. Reine Angew. Math., 1979, **310**: 1–6.

[2] CASTAING C and VALADIER M. *Convex analysis and measurable multifunctions* [M]. L.N.M., Vol.580, Springer-Verlag, Berlin and New York, 1977.

[3] DONG W L and WANG Z P. *On representation and regularity of continuous parameter multivalued martingales* [J]. Proceedings of the American Mathematical Society, 1998, **126**(6): 1799–1810.

[4] HE S W, WANG J G and YAN J A. *Semimartingale theory and stochastic Calculus* [M]. Science Press and CRC Press Inc. Boca Raton Ann Arbor London Tokyo, 1992.

[5] HESS C. *On multivalued martingales whose values may be unbounded: martingale selectors and Mosco convergence* [J]. J. Multivariate Anal., 1991, **39**: 175–201.

[6] HIAI F. *Convergence of conditional expectations and strong laws of large numbers for multivalued radom variables* [J]. Trans. Amer. Math. Soc., 1985, **291**: 613–627.

[7] HIAI F and UMEGAKI H. *Integrals, conditional expectation and martingales of multivalued functions* [J]. J. Multivariate Anal., 1977, **7**: 149–182.

[8] LUU D Q. *Quelques resultats de representatipon des amarts uniformes multivoques* [J]. C. R. A. S. Paris, 1985, **300**: 63–65.

[9] PAPAGEORGIOU N S. *On the theory of Banach space valued multifunctions (1) Integration and conditional expectations* [J]. J. Multivariate. Anal., 1985, A, **17**: 185–206.

[10] PAPAGEORGIOU N S. *On the conditional expection and convergence properties of random sets* [J]. Trans. Amer. Math. Soc., 1995, **347**: 2495–2515.

[11] WANG R M and WANG Z P. *Set-valued Stationary Processes* [J]. Journal of Multivariate Analysis, 1997, **63**(1): 190–198.

[12] WANG R M and WANG Z P. *Doob's Stopping theorems for set-valued (super, sub) martingales with discrete parameters* [J]. Chinese Journal of Applied Probability and Statistics, 1998, **14**(2):

[12] ZHANG W X, WANG Z P and GAO Y. *Set-valued stochastic processes* (in Chinese) [M]. Science Press, Beijing, 1996.

连续时间集值 (上、下) 鞅 Doob- 停时定理

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摘 要: 本文首先给出了 J_L 意义下的集值鞅的正则性，然后证明了几种连续时间集值 (上、下) 鞅 Doob- 停时定理.