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李群方法对一阶偏微分方程的应用*

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摘 要：本文以李群为工具，给出了一种将一阶非线性偏微分方程化简为一阶拟线性方程或可积的一阶拟线性方程的方法. 该方法可用于某些两个自变元的，接受一个或两个李群的一阶非线性偏微分方程，特别可用于某些单自由度 Lagrange 系统的 Hamilton-Jacobi 方程的求解.

关键词：李群；一阶偏微分方程；Hamilton-Jacobi 方程；可积性.

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1 引 言

近年来，李群方法在微分方程的研究中有着非常广泛的应用(文[1－4]). 根据李群理论，如果常微分方程(组)接受一个李群，则一般可通过坐标变换将该方程(组)降低一阶(文[5]，[6]). 但如果偏微分方程(组)接受一个李群，则一般只能通过坐标变换使方程不显含某个变元. 在特殊情况下，有可能求出方程(组)的某些特殊解. 在极特殊的情况下，有可能对方程(组)进行线性化. 本文试图对于李群在两个自变元的一阶偏微分方程上的应用做一点探讨，在不太强的条件下，将方程化简为一阶拟线性方程或可积的一阶偏微分方程. 该方法可用于某些单自由度 Lagrange 系统的 Hamilton-Jacobi 方程的求解.

2 主要结果

首先引进一阶偏微分方程

$$F(x,y,u,\frac{\partial u}{\partial x},\frac{\partial u}{\partial y}) = 0 \tag{1}$$

接受李群的定义(详细内容见文[5]，[6]). 设

$$V = A_1(x,y,u)\frac{\partial}{\partial x} + A_2(x,y,u)\frac{\partial}{\partial y} + A_3(x,y,u)\frac{\partial}{\partial u},$$
$$(A_1(x,y,u),A_2(x,y,u),A_3(x,y,u)) \in C^\infty(\Omega),\Omega \subset R^3).$$

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定义 V 的一次扩展为

$$V^{(1)}=A_1(x,y,u)\frac{\partial}{\partial x}+A_2(x,y,u)\frac{\partial}{\partial y}+A_3(x,y,u)\frac{\partial}{\partial u}+$$

$$\left(\frac{\partial A_3}{\partial x}-\frac{\partial A_1}{\partial x}u_x-\frac{\partial A_2}{\partial x}u_y\right)\frac{\partial}{\partial u_x}+\left(\frac{\partial A_3}{\partial y}-\frac{\partial A_1}{\partial y}u_x-\frac{\partial A_2}{\partial y}u_y\right)\frac{\partial}{\partial u_y}.$$

若当 $F(x,y,u,u_x,u_y)=0$ 时有 $V^{(1)}F(x,y,u,u_x,u_y)=0$，则称方程 $F=0$ 接受以 V 为无穷小生成元的李群.

设方程(1)接受以 V 为无穷小生成元的李群，若能找出函数 $\varphi_1(x,y,u),\varphi_2(x,y,u),\varphi_3(x,y,u)\in C^\infty(\Omega)$，使得

$$V\varphi_1=V\varphi_2=0,V\varphi_3=1, \tag{2}$$

且 $\varphi_1,\varphi_2,\varphi_3$ 函数无关. 则方程(1)经过

$$x_1=\varphi_1(x,y,u),y_1=\varphi_2(x,y,u),u_1=\varphi_3(x,y,u) \tag{3}$$

变换之后不显含 u_1 (文[5],[6])，所以方程(1)变为

$$F_1\left(x_1,y_1,\frac{\partial u_1}{\partial x_1},\frac{\partial u_1}{\partial y_1}\right)=0. \tag{4}$$

设能从方程(4)中解出 $\frac{\partial u_1}{\partial x_1}$ 或 $\frac{\partial u_1}{\partial y_1}$，不妨设有

$$\frac{\partial u_1}{\partial x_1}=f\left(x_1,y_1,\frac{\partial u_1}{\partial y_1}\right). \tag{5}$$

令 $U=\frac{\partial u_1}{\partial y_1}$，则有

$$\frac{\partial U}{\partial x_1}=\frac{\partial}{\partial y_1}\left(\frac{\partial u_1}{\partial x_1}\right)=\frac{\partial}{\partial y_1}(f(x_1,y_1,U(x_1,y_1)))=\frac{\partial f(x_1,y_1,U)}{\partial y_1}+\frac{\partial f(x_1,y_1,U)}{\partial U}\frac{\partial U}{\partial y_1}, \tag{6}$$

因此方程(1)被化为一阶拟线性方程(6). 若能求出(6)的解 $U=g(x_1,y_1)$，则有

$$u_1=\int g(x_1,y_1)\mathrm{d}y_1+g_1(x_1). \tag{7}$$

由(5)有 $\frac{\partial u_1}{\partial x_1}=\frac{\partial}{\partial x_1}\int g(x_1,y_1)\mathrm{d}y_1+g_1'(x_1)=f(x_1,y_1,g(x_1,y_1))$，所以

$$g_1'(x_1)=f(x_1,y_1,g(x_1,y_1))-\frac{\partial}{\partial x_1}\int g(x_1,y_1)\,\mathrm{d}y_1=g_2(x_1).$$

(由 g 满足(6)易见 $f(x_1,y_1,g(x_1,y_1))-\frac{\partial}{\partial x_1}\int g(x_1,y_1)\,\mathrm{d}y_1$ 与 y_1 无关). 由(7)可得

$$u_1=\int g(x_1,y_1)\,\mathrm{d}y_1+\int g_2(x_1)\,\mathrm{d}x_1+c.$$

将(3)代入上式即得方程(1)的解. 综合上述结果，可得下面的

定理 1 若方程(1)接受一个李群(其无穷小生成元为 V)，且能找到 $C^\infty(\Omega)$ 中函数无关的函数 $\varphi_1(x,y,u),\varphi_2(x,y,u),\varphi_3(x,y,u)$，使得(2)式成立，则由变换(3)可将方程(1)化为(4). 若能从(4)中解出 $\frac{\partial u_1}{\partial x_1}$ (或 $\frac{\partial u_1}{\partial y_1}$)则可将(1)的求解问题化成(6)的求解问题.

下面考虑方程接受两个李群的情况. 设方程(1)接受两个李群，它们的无穷小生成元分别为

$$V_1=A_1(x,y,u)\frac{\partial}{\partial x}+A_2(x,y,u)\frac{\partial}{\partial y}+A_3(x,y,u)\frac{\partial}{\partial u},$$

$$V_2 = B_1(x,y,u)\frac{\partial}{\partial x} + B_2(x,y,u)\frac{\partial}{\partial y} + B_3(x,y,u)\frac{\partial}{\partial u},$$

其中 $\begin{vmatrix} A_1 & A_2 \\ B_1 & B_2 \end{vmatrix}, \begin{vmatrix} A_2 & A_3 \\ B_2 & B_3 \end{vmatrix}, \begin{vmatrix} A_3 & A_1 \\ B_3 & B_1 \end{vmatrix}$ 不全为零，且 $[V_1,V_2] = V_1V_2 - V_2V_1 = 0$. 设能解出函数无关的函数 $\varphi_1(x,y,u), \varphi_2(x,y,u), \varphi_3(x,y,u) \in C^\infty(\Omega)$，使得

$$V_1\varphi_1 = V_2\varphi_2 = 1, V_1\varphi_2 = V_1\varphi_3 = V_2\varphi_1 = V_2\varphi_3 = 0. \tag{8}$$

做变换

$$x_1 = \varphi_1(x,y,u), y_1 = \varphi_3(x,y,u), u_1 = \varphi_2(x,y,u), \tag{9}$$

则变换后的方程不显含 x_1 和 u_1. 方程(1)变为

$$F_2(y_1, \frac{\partial u_1}{\partial x_1}, \frac{\partial u_1}{\partial y_1}) = 0. \tag{10}$$

若能从(10)中解出 $\frac{\partial u_1}{\partial y_1} = f(y_1, \frac{\partial u_1}{\partial x_1})$，令 $U = \frac{\partial u_1}{\partial x_1}$，则有

$$\frac{\partial U}{\partial y_1} = \frac{\partial}{\partial x_1} f(y_1, U) = \frac{\partial f}{\partial U}\frac{\partial U}{\partial x_1}. \tag{11}$$

该方程可用特征线方法求解. 由此可得下面的

定理 2 设方程(1)接受两个李群，它们的无穷小生成元 V_1, V_2 满足 $[V_1,V_2] = 0$，且

$$\begin{vmatrix} A_1 & A_2 \\ B_1 & B_2 \end{vmatrix}, \begin{vmatrix} A_2 & A_3 \\ B_2 & B_3 \end{vmatrix}, \begin{vmatrix} A_3 & A_1 \\ B_3 & B_1 \end{vmatrix}$$

不全为零. 若能找出函数无关的 $\varphi_1(x,y,u), \varphi_2(x,y,u), \varphi_3(x,y,u) \in C^\infty(\Omega)$，使得(8)式成立，则在变换(9)下方程(1)可化为方程(10). 若能从(10)中解出 $\frac{\partial u_1}{\partial y_1}$，则可将方程(1)化为可积的一阶拟线性齐次方程(11).

3 应用举例

例 1 考虑非线性方程

$$\frac{\partial u}{\partial x} = f_1(x, u, \frac{\partial u}{\partial y}), \tag{12}$$

该方程接受以 $V = \frac{\partial}{\partial y}$ 为无穷小生成元的李群. 取 $\varphi_1 = x, \varphi_2 = u, \varphi_3 = y$，则有(2)式成立. 显然 $\varphi_1, \varphi_2, \varphi_3$ 函数无关. 做变换 $x_1 = \varphi_1 = x, y_1 = \varphi_2 = u, u_1 = \varphi_3 = y$，则有

$$\begin{aligned} dy_1 = du &= \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy = \frac{\partial u}{\partial x}dx_1 + \frac{\partial u}{\partial y}du_1 \\ &= \frac{\partial u}{\partial x}dx_1 + \frac{\partial u}{\partial y}(\frac{\partial u_1}{\partial x_1}dx_1 + \frac{\partial u_1}{\partial y_1}dy_1), \end{aligned}$$

即 $(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial u_1}{\partial x_1})dx_1 + (\frac{\partial u}{\partial y}\frac{\partial u_1}{\partial y_1} - 1)dy_1 = 0$，所以

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial u_1}{\partial x_1} = 0, \frac{\partial u}{\partial y}\frac{\partial u_1}{\partial y_1} = 1.$$

从上式解出 $\frac{\partial u}{\partial x}$ 及 $\frac{\partial u}{\partial y}$，则方程(12)化为 $\frac{\partial u_1}{\partial x_1} = -\frac{\partial u_1}{\partial y_1}f_1(x_1, y_1, (\frac{\partial u_1}{\partial y_1})^{-1})$. 令 $U = \frac{\partial u_1}{\partial y_1}$，由前述方

法可将方程(12)化为一阶拟线性方程

$$\frac{\partial U}{\partial x_1}+\frac{\partial}{\partial U}(Uf_1(x_1,y_1,U^{-1}))\frac{\partial U}{\partial y_1}=-U\frac{\partial}{\partial y_1}f_1(x_1,y_1,U^{-1}).$$

例 2 考虑方程

$$u_y=f_2(u,\frac{u_x}{u_y})\tag{13}$$

分别接受以 $V_1=\frac{\partial}{\partial x}$ 及 $V_2=\frac{\partial}{\partial y}$ 为无穷小生成元的两个李群，且 $[V_1,V_2]=0$，取 $\varphi_1=x,\varphi_2=y,\varphi_3=u$，则 $V_1,V_2,\varphi_1,\varphi_2,\varphi_3$ 满足定理 2 中的要求，做变换 $x_1=\varphi_1=x,y_1=\varphi_3=u,u_1=\varphi_2=y$，则由例 1 中的推导，方程(13)变为 $\frac{\partial u_1}{\partial y_1}=(f_2(y_1,-\frac{\partial u_1}{\partial x_1}))^{-1}$. 令 $U=\frac{\partial u_1}{\partial x_1}$，则可将(13)化为

$$\frac{\partial U}{\partial y_1}-\frac{\partial}{\partial U}(f_2(y_1,-U))^{-1}\frac{\partial U}{\partial x_1}=0.$$

该方程可用特征线方法求解.

例 3 考虑方程

$$yu\frac{\partial u}{\partial y}-y^2(\frac{\partial u}{\partial y})^2+f_3(\frac{u}{y})(\frac{\partial u}{\partial x})^2=0\tag{14}$$

接受以 $V_1=\frac{\partial}{\partial x}$ 及 $V_2=y\frac{\partial}{\partial y}+u\frac{\partial}{\partial u}$ 为无穷小生成元的两个李群，且 $[V_1,V_2]=0$. 不妨设 $y>0$. 取 $\varphi_1=x,\varphi_2=\ln y,\varphi_3=\frac{u}{y}$，则(8)式成立. 做变换(9)并令 $U=\frac{\partial u_1}{\partial x_1}$，则可将方程(14)化为

$$\frac{\partial U}{\partial y_1}-\frac{2}{y_1}f_3(y_1)U\frac{\partial U}{\partial x_1}=0.$$

该方程显然可解.

例 4 考虑广义 Duffing 方程

$$\frac{d^2x}{dt^2}+Q_0t^{-\frac{3}{2}}+Q_1xt^{-2}+Q_2x^2t^{-\frac{5}{2}}+Q_3x^3t^{-3}=0,\tag{15}$$

其中 Q_i 为任意常数，$i=0,1,2,3$(见文[7]).

该方程的 Hamilton-Jacobi 方程为

$$\frac{\partial S}{\partial t}+\frac{1}{2}(\frac{\partial S}{\partial x})^2+Q_0xt^{-\frac{3}{2}}+\frac{1}{2}Q_1x^2t^{-2}+\frac{1}{3}Q_2x^3t^{-\frac{5}{2}}+\frac{1}{4}Q_3x^4t^{-3}=0.\tag{16}$$

(16)接受以 $V_1=t\frac{\partial}{\partial t}+\frac{1}{2}x\frac{\partial}{\partial x}$ 及 $V_2=\frac{\partial}{\partial S}$ 为无穷小生成元的两个李群，且 $[V_1,V_2]=0$，取 $\varphi_1=\ln t,\varphi_2=S-\frac{x^2}{4t},\varphi_3=xt^{-\frac{1}{2}}$，经变换(9)后方程(16)变为

$$\frac{\partial u_1}{\partial x_1}+\frac{1}{2}(\frac{\partial u_1}{\partial y_1})^2+Q_0y_1+\frac{1}{2}(Q_1-\frac{1}{4})y_1^2+\frac{1}{3}Q_2y_1^3+\frac{1}{4}Q_3y_1^4=0.$$

令 $U=\frac{\partial u_1}{\partial x_1}$，可得

$$\frac{\partial U}{\partial y_1}=\frac{\partial}{\partial x_1}(\frac{\partial u_1}{\partial y_1})=\frac{\pm\frac{\partial U}{\partial x_1}}{\sqrt{-2(U+Q_0y_1+\frac{1}{2}(Q_1-\frac{1}{4})y_1^2+\frac{1}{3}Q_2y_1^3+\frac{1}{4}Q_3y_1^4)}}.$$

该方程可用特征线方法求解(文[7]用其它方法得出了(16)的解). 利用 Hamilton-Jacobi 定理则可求出方程(15)的通解.

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The Application of Lie Group Method to First Order Partial Differential Equations

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Abstract: A way of transforming first order nonlinear PDE to first order quasilinear PDE or integrable first order PDE by using Lie group method is presented. The presented way can be applied to some first order nonlinear PDE's which admit one or two Lie groups and have two independent variables, especially to Hamilton-Jacobi equations of some Lagrange systems which have one degree of freedom.

Key words: Lie group; first order PDE; Hamilton-Jacobi equation; integrability.