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On a Property of Roots of Polynomials *

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Abstract: In [1], a property of roots of polynomials is considered, which involves the existence of local analytic solutions of polynomial-like functional iterative equations. In this paper we discuss this property and obtain a succinct condition to decide whether this property holds. Our main result is: A polynomial $\lambda_n z^n + \cdots + \lambda_2 z^2 + \lambda_1 z + \lambda_0$ of degree n has a root α such that $\inf\{|\lambda_n \alpha^{nm} + \cdots + \lambda_2 \alpha^{2m} + \lambda_1 \alpha^m + \lambda_0| : m = 2, 3, \cdots\} > 0$ if and only if at least one of the following two conditions holds: (i) the polynomial has a root β satisfying $|\beta| > 1$; (ii) the polynomial has a root β satisfying $|\beta| < 1$, and $\lambda_0 \neq 0$.

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1. Introduction

Let $n \geq 1, \lambda_n \neq 0$. In [1] the author discussed the functional iterative equation

$$\lambda_n f^n(z) + \cdots + \lambda_2 f^2(z) + \lambda_1 f(z) = F(z), \tag{1}$$

and obtained the following

Theorem A *Let r_1 be a positive number. Suppose $F(z)$ is analytic in $|z| < r_1$, and its power series expansion is $F(z) = -\lambda_0 z + \sum_{m=2}^{\infty} c_m z^m$. If the following Property Q holds, then there exists $r > 0$ such that the equation (1) has an analytic solution in $|z| < r$.*

Property Q *The polynomial $\lambda_n z^n + \cdots + \lambda_2 z^2 + \lambda_1 z + \lambda_0$ has a root α satisfying $\inf\{|\lambda_n \alpha^{nm} + \cdots + \lambda_2 \alpha^{2m} + \lambda_1 \alpha^m + \lambda_0| : m = 2, 3, \cdots\} > 0$.*

However, there is a problem which has not been discussed in [1]: In what case does Property Q hold? Can we find a simple rule to decide whether Property Q holds or not? In this note, we will discuss this problem and obtain an answer.

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2. Decision on Property Q

Let $\mathbf{C}$ be the complex plane, and $P : \mathbf{C} \to \mathbf{C}$ be a map. If for some integer $n \geq 1$, there exist complex numbers $\lambda_0, \lambda_1, \cdots, \lambda_n$ with $\lambda_n \neq 0$ such that

$$P(z) = \lambda_n z^n + \cdots + \lambda_2 z^2 + \lambda_1 z + \lambda_0, \qquad \text{for all } z \in \mathbf{C}, \tag{2}$$

then P is called a polynomial function [2] of degree n, or simply, a polynomial. In general, $\lambda_n z^n + \cdots + \lambda_2 z^2 + \lambda_1 z + \lambda_0$ is also called a polynomial, but in this case we must note to clarify that $\lambda_n z^n + \cdots + \lambda_2 z^2 + \lambda_1 z + \lambda_0$ actually represents the whole map $P : \mathbf{C} \to \mathbf{C}$ or only the value $P(z)$ of the map at a point $z \in \mathbf{C}$. Let $w \in \mathbf{C}$. w is called a root of the polynomial P (or a zero point of P) if $P(w) = 0$. It is well-known that a polynomial of degree n has just n roots (where a k-ple root is regarded as k roots).

For any $W \subset \mathbf{C}$, denote by $\overline{W}$ or $\mathrm{Cl}(W)$ the closure of W. Noting $\lim_{|z|\to\infty} |P(z)| = \infty$, by the continuity of P we can easily verify the following

Lemma 1 *Let $P : \mathbf{C} \to \mathbf{C}$ be a polynomial of degree $n \geq 1$, $\alpha_1, \alpha_2, \cdots, \alpha_n$ be the n roots of P, and W be a nonempty subset of $\mathbf{C}$. Then $\inf\{|P(z)| : z \in W\} = 0$ if and only if $\overline{W} \cap \{\alpha_1, \alpha_2, \cdots, \alpha_n\} \neq \emptyset$.* □

Lemma 2 *Let $\alpha \in S^1$ ($\equiv \{z \in \mathbf{C} : |z| = 1\}$), and let $j \geq 1$ be an integer. Then*

$$\alpha \in \mathrm{Cl}(\{\alpha^{jm+1} : m = 1, 2, \cdots\}). \tag{3}$$

Proof We may assume $\alpha = e^{2\pi i r}$, where $r \in [0, 1)$, $i = \sqrt{-1}$. If r is a rational number, i.e. if $r = l/k$ for some integers $k > l \geq 0$, then (3) holds since $\alpha = \alpha^{kj+1}$.

If r is an irrational number, then jr is also irrational. Define $f : S^1 \to S^1$ by

$$f(e^{2\pi i x}) = e^{2\pi i (x + jr)}, \qquad \text{for any } x \in R.$$

Then f is an irrational rotation. By [3], every orbit of f is dense in S^1. Let $\beta = \alpha^{j+1}$. Noting that the orbit $O(\beta, f)$ ($\equiv \{\beta, f(\beta), f^2(\beta), \cdots\}) = \{\alpha^{j+1}, \alpha^{2j+1}, \alpha^{3j+1}, \cdots\}$, we see that (3) still holds. □

Now we present the main result of this paper:

Theorem 1 *Let $P : \mathbf{C} \to \mathbf{C}$ be a polynomial of degree $n \geq 1$ defined as in (2). Then Property Q above holds if and only if at least one of the following two conditions holds:*

(C.1) *P has a root α satisfying $|\alpha| > 1$;*

(C.2) *P has a root α satisfying $|\alpha| < 1$, and $\lambda_0 \neq 0$.*

Proof Let $\alpha_1, \cdots, \alpha_n$ be the n roots of P. We may assume $|\alpha_1| \leq |\alpha_2| \leq \cdots \leq |\alpha_n|$.

Sufficiency If (C.1) holds, take $\alpha = \alpha_n$, then $1 < |\alpha| < |\alpha^2| < |\alpha^3| < \cdots$. Let $W = \{\alpha^m : m = 2, 3, \cdots\}$. Then $\overline{W} \cap \{\alpha_1, \alpha_2, \cdots, \alpha_n\} = W \cap \{\alpha_1, \alpha_2, \cdots, \alpha_n\} = \emptyset$. By Lemma 1, $\inf\{|P(\alpha^m)| : m = 2, 3, \cdots\} > 0$. Noting that $P(\alpha^m) = \lambda_n \alpha^{nm} + \cdots + \lambda_2 \alpha^{2m} + \lambda_1 \alpha^m + \lambda_0$, we see that Property Q holds.

If (C.2) holds, take $\alpha = \alpha_1$, then $1 > |\alpha| > |\alpha^2| > |\alpha^3| > \cdots > 0$. Let $W = \{\alpha^m : m = 2, 3, \cdots\}$. Then $\overline{W} = W \cup \{0\}$, and $\overline{W} \cap \{\alpha_1, \alpha_2, \cdots, \alpha_n\} = \emptyset$. Hence, by Lemma 1,

$\inf\{|P(\alpha^m)| : m = 2, 3, \cdots\} > 0$, i.e. Property Q still holds.

Necessity Write $B^2 = \{z \in \mathbf{C} : |z| \leq 1\}$. If neither (C.1) nor (C.2) is true, then just one of the following two conditions is true:

(C.3) $\{\alpha_1, \cdots, \alpha_n\} \subset S^1$;

(C.4) $\{\alpha_1, \cdots, \alpha_n\} \subset B^2$, and $\alpha_1 = 0$.

By Lemma 2, no matter whether (C.3) or (C.4) is true, we have $\mathrm{Cl}(\{\alpha_k^m : m = 2, 3, \cdots\}) \cap \{\alpha_1, \alpha_k\} \neq \emptyset$ for each $k \in \{1, 2, \cdots, n\}$. Thus, by Lemma 1, Property Q does not hold. The proof of Theorem 1 is completed. □

From Theorem 1 we obtain the following theorem, of which the conditions are relatively succinct and clear, and by which we can replace Theorem A.

Theorem 2 *Let $P : \mathbf{C} \to \mathbf{C}$ be a polynomial of degree $n \geq 1$ defined as in (2), and r_1 be a positive number. Suppose $F(z)$ is analytic in $|z| < r_1$, and its power series expansion is $F(z) = -\lambda_0 z + \sum_{m=2}^{\infty} c_m z^m$. If one of the following two conditions holds:*

(C.1) P has a root β satisfying $|\beta| > 1$;

(C.2) P has a root β satisfying $|\beta| < 1$, and $\lambda_0 \neq 0$,

then there exists $r > 0$ such that the equation (1) has an analytic solution in $|z| < r$.

Let $j \geq 2$ be a given integer. We can also consider the following

Property $\mathrm{Q_j}$ *The polynomial $\lambda_n z^n + \cdots + \lambda_2 z^2 + \lambda_1 z + \lambda_0$ has a root α satisfying $\inf\{|\lambda_n \alpha^{n(jm+1)} + \cdots + \lambda_2 \alpha^{2(jm+1)} + \lambda_1 \alpha^{jm+1} + \lambda_0| : m = 1, 2, \cdots\} > 0$.*

Proposition 1 *Let $P : \mathbf{C} \to \mathbf{C}$ be a polynomial of degree $n \geq 1$ defined as in (2). Then Property Q_j holds if and only if the above condition (C.1) or (C.2) holds.*

The proof of Proposition 1 is similar to that of Theorem 1, and is omitted.

Since $\{\alpha^{jm+1} : m = 1, 2, 3, \cdots\} \subset \{\alpha^m : m = 2, 3, 4, \cdots\}$, Property $\mathrm{Q_j}$ seems weaker than Property Q. However, by Theorem 1 and Proposition 1, these two properties are actually equivalent.

3. Decision on conditions (C.1) and (C.2)

One has found various algorithms for computing roots of polynomials [4]. Thus, in principle, there are no difficulties in deciding whether the condition (C.1) or (C.2) in Theorem 1 holds or not when $\lambda_0, \lambda_1, \cdots, \lambda_n$ are concrete numbers. Furthermore, By means of some relations between roots of polynomials and their coefficients, we can obtain several succinct results which can be conveniently used to judge conditions (C.1) and (C.2) directly from the coefficients $\lambda_0, \lambda_1, \cdots, \lambda_n$ without computing the roots. For example, we have the following proposition.

Proposition 2 *Let $P : \mathbf{C} \to \mathbf{C}$ be a polynomial of degree $n \geq 1$ defined as in (2).*

(i) If $|\lambda_0| > |\lambda_n|$, then condition (C.1) holds.

In general, let $k \in \{1, \cdots, n\}$. If $\lambda_j = 0$ for $0 \leq j < n - k$, and there exists some $m \in \{1, \cdots, k\}$ such that $|\lambda_{n-m}/\lambda_n| > k!/[m!(k-m)!]$, then condition (C.1) holds.

(ii) If $0 < |\lambda_0| < |\lambda_n|$, then condition (C.2) holds.

(iii) If n is odd, $\lambda_0, \lambda_1, \cdots, \lambda_n$ are all real numbers, $\lambda_0 \neq 0$, $\sum_{j=0}^{n} \lambda_j \neq 0$, and

$\sum_{j=0}^{n}(-1)^j\lambda_j \neq 0$, *then at least one of Conditions (C.1) and (C.2) holds.*

Proof (i) Suppose $\beta_1, \beta_2, \cdots, \beta_n$ are the n roots of P with $|\beta_1| \geq |\beta_2| \geq \cdots \geq |\beta_n|$.Then $|\beta_{k+1}| = \cdots = |\beta_n| = 0$ since $\lambda_0 = \lambda_1 = \cdots = \lambda_{n-k-1} = 0$. If $|\beta_1| \leq 1$, then, for every $m \in \{1, \cdots, k\}$, $|\lambda_{n-m}/\lambda_n| = |\sum_{1\leq j_1<\cdots<j_m\leq k}\beta_{j_1}\cdots\beta_{j_m}| \leq k!/[m!(k-m)!]$. This contradicts the condition of the proposition. Thus $|\beta_1| > 1$, i.e. (C.1) holds.

(ii) Let $\alpha_1, \cdots, \alpha_n$ be the n roots of P. Then $\min\{|\alpha_1|, \cdots, |\alpha_n|\} < 1$ since $\prod_{m=1}^{n}|\alpha_m| = |\lambda_0/\lambda_n| < 1$. In addition, we have $\lambda_0 \neq 0$. Hence (C.2) holds.

(iii) It is easy to see that P has at least a real root $r \notin \{-1, 0, 1\}$. Thus at least one of (C.1) and (C.2) holds. □

References:

[1] SI Jian-guo. *Existence of local analytic solutions of the iterative equation* $\sum_{i=1}^{n}\lambda_i f^i(x) = F(x)$ [J]. Acta. Math. Sinica, 1994, **37**(5): 590–600.

[2] MAI Jie-hua. *Regional stability of zero points of families of polynomial functions* [J]. Chin. J. of Contemporary Math., 1995, **16**(4): 415-427; Chin. Ann. of Math (A)., 1995, **16**(5): 604–613.

[3] NITECKI Z. *Differentiable Dynamics* [M]. M.I.T.Press, London, 1971.

[4] WANG Ze-ke. *Fixed Point Algorithms* [M]. Shanghai Scientific and Technical Publishers, Shanghai, 1987.

多项式的根的一个性质的判定

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摘 要: 文献 [1] 在讨论多项式型的函数迭代方程的局部解析解的存在性时涉及到了多项式的根的一个性质. 本文给出了判定该性质是否成立的一个简洁的条件，证明了多项式 $\lambda_n z^n + \cdots + \lambda_2 z^2 + \lambda_1 z + \lambda_0$ 有一个根 α 满足 $\inf\{|\lambda_n\alpha^{nm} + \cdots + \lambda_2\alpha^{2m} + \lambda_1\alpha^m + \lambda_0| : m = 2, 3, \cdots\} > 0$ 当且仅当如下两个条件之中至少有一个成立： (i) 该多项式有一个根 β 满足 $|\beta| > 1$; (ii) 该多项式有一个根 β 满足 $|\beta| < 1$, 且 $\lambda_0 \neq 0$.