Journal of Mathematical Research & Exposition
Vol.21, No.1, 77-80, February, 2001

Mistake in the Paper "The Generalization of Whitney's Lemma and Application" *

CAO Yi
(Dept. of Math., Guizhou University, Guiyang 550025, China)

Abstract: We find that there is a serious mistake in the proof of Lemma 3 of paper [1]. The author wants to prove, copying the original method in [2], the global Borel theorem. But a simple example shows that this is impossible.

Key words: Borel theorem; function-germ; convergence uniformly.

Classification: AMS(1991) 58C,32S20,73E05/CLC O186.33,O334.1

Document code: A **Article ID**: 1000-341X(2001)01-0077-04

The Lemma 3 (Generalization of E.Borel Theorem) in paper [1] is (see [1], p321): Given a sequence of C^∞ function on $R^n\{f_m(x)\}(m=0,1,2,\cdots)$, there exists a C^∞ function on $R^n\times R\to R$ such that $D^{0,m}f(x,y)|_{R^n\times\{0\}}=f_m(x)$.

We extract the first part of the proof in [1] as follows:

Proof $\forall r>0$ and $f_m(x)$, we can make a C^∞ function $\tilde{f}_m$ with compact support $K(0,r)$ which is the closed ball around 0 with radius $r:|x|\le r$, and $\tilde{f}_m(x)=f_m(x)$ on $K(0,\frac{r}{2})$. For simple and convennient, $\tilde{f}_m(x)$ is written as $f_m(x)$ yet. Once more take a C^∞ function

$$\varphi:R\to R, 0\le\varphi\le 1, \varphi(y)=1 \text{ for all } |y|\le\frac{r}{2}, \varphi(y)=0 \text{ for } |y|\ge r.$$

Let

$$f(x,y)=\sum_{m=0}^{\infty}\frac{f_m(x)}{m!}y^m\varphi(t_my). \tag{1}$$

Assume that the sequence $t_m(m=0,1,2,\cdots)$ can be defined to make the series

$$\sum_{m=0}^{\infty}D^{\alpha,p}(\frac{f_m(x)}{m!}y^m\varphi(t_my)) \tag{2}$$

convergent uniformly for each $\alpha=(\alpha_1,\alpha_2,\cdots,\alpha_n)$ and each natural number p, then $f(x,y)$ would be a C^∞ function and could be differentated term by term.

*Received date: 2000-03-29
Biography: CAO Yi (1942-), male, Professor.

Since for each $t_m, \varphi(t_m y) = 1$, as long as $|y|$ is small enough, and the differentiation is a local property. This gives

$$D^{0,m} f(x,y)|_{R^n \times \{0\}} = f_m(x).$$

We know that when $f_m(x)(m = 0,1,2,\cdots)$ are germs, by the proof of Borel Theorem (see [2], p32), (1) defines an C^∞ germ $f(x,y) : (R^n \times R^k, \{0\} \times \{0\}) \to R$, which satifies the conclusion of Borel Theorem. The paper [1], copying the method of [2], wants to get a global Borel theorem. But following example shows this is impossible by the same method.

To show what the mistake of [1] is, we should introduce several important concepts. In analysis the conception that a function sequence $\{f_n(x)\}$ converges uniformly on a domain D, is familiar with us. But to a function-germ sequence its convergence uniformly has special meaning, since the germ is an infinitesimal concept which shows a dynamic state. Here we give the definition of function-germ, firstly. Following functions and function-germs we mentioned all are C^∞.

Definition 1 *Let $a \in R^n$, and U_1, U_2 be two neighborhoods of $a, f_1 : U_1 \to R$ and $f_2 : U_2 \to R$ be two functions. If there is an open neighborhood U of a such that*

$$U \subset U_1 \cap U_2 \quad \text{and} \quad f_1|_U = f_2|U,$$

we say f_1 and f_2 are equivalent. The equivalence class of function f is called a germ of f at a, denoted as $\tilde{f} : (R^n : a) \to R$. Where (R^n, a) is called the domain-germ which is also an equivalence class: generally, let $D \subset R^n$, two neighborhoods U and V of D are equivalent, if there is an open neighborhood W of D, such that $W \subset U \cap V$. In other words, topologically, (R^n, D) is a neighborhood system of D.

Although function and function-germ have the similar express of symbol, their meanings are different. Every representative of a function-germ is a common function.

We denote a function-germ $\tilde{f}$ as f, and $(R^n \times R, R^n \times \{a\})$ as $R^n \times \{a\}$ sometimes, when there isn't any confusion occurring.

Definition 2 *Let $\{f_n(x,y)\}$ be a function-germ sequence on domain-germ $(R^n \times R, R^n \times \{a\})$ we say that $\{f_n(x,y)\}$ converges uniformly to a function-germ $f(x,y)$ on $(R^n \times R, R^n \times \{a\})$, if for any $\varepsilon > 0$ there is a natural number N and an open neighborhood U_ε of $R^n \times \{a\}$ in $R^n \times R$ (and a representative $f \in \tilde{f}$) such that, when $n > N$ (there exists a representative $f_n \in \tilde{f}$) the inequality*

$$|f_n(x,y) - f(x,y)| < \varepsilon$$

holds for all $(x,y) \in U_\varepsilon$.

Remark Below, like the words in the brackets above will no longer be emphasized.

We may also use the Cauchy's principle to represent Definition 2: function-germ sequence $\{f_n(x,y)\}$ converges uniformly on $(R^n \times R, R^n \times \{a\})$, if and only if, for any given $\varepsilon > 0$ there is a natural number N and an open neighborhood U_ε of $R^n \times \{a\}$, such that, when $p > N$ and $q > N$ the inequality

$$|f_p(x,y) - f_q(x,y)| < \varepsilon \qquad (\triangle)$$

holds for all $(x, y) \in U_\varepsilon$.

From Cauchy's principle we get the following result, directly.

Lemma *Let $\{f_n(x,y)\}$ be a function-germ sequence on $(R^n \times R, R^n \times \{a\})$. If there is a number $\varepsilon_0 > 0$ for any open neighborhood U of $R^n \times \{a\}$, and any natural number N, there always exists an $(\bar{x}, \bar{y}) \in U$ and two natural number p and q with $p \geq N$ and $q \geq N$, such that*

$$|f_p(\bar{x}, \bar{y}) - f_q(\bar{x}, \bar{y})| \geq \varepsilon_0. \qquad (\triangle\triangle)$$

Then $\{f_n(x,y)\}$ can not be convergent uniformly on $(R^n \times R, R^n \times \{a\})$.

Example Let $f_m(x) = m! x^m (x \in R, m = 0, 1, 2, \cdots)$. Then (1) becomes:

$$f(x,y) = \sum_{m=0}^{\infty} (xy)^m \varphi(t_m y). \qquad (3)$$

Set

$$f_n(x,y) = \sum_{m=0}^{n} (xy)^m \varphi(t_m y) \qquad (4)$$

We will show that the sequence above can not converge uniformly for any $\{t_m\}$ on the domain-germ $(R \times R, R \times \{0\})$. Suppose sequence $\{t_m\}$ is given. It is clear that we can assume $t_m \neq 0, \forall m$. For any natural number m, choose y_m with $0 < |y_m| \leq \min(\frac{1}{2|t_m|}, \frac{1}{m})$, and $x_m = \frac{1}{y_m}$. For any open neighborhood $R \times U$ of $R \times \{0\}$, when m is big enough, there is $(x_m, y_m) \in R \times U$ and

$$(x_m y_m)^m \varphi(t_m y_m) = 1.$$

So

$$|f_m(x_m, y_m) - f_{m-1}(x_m, y_m)| = 1.$$

It shows, by Lemma, that the sequence (4) can not converge uniformly on the domain-germ $(R \times R, R \times \{0\})$. This example will overthrow the proof of Lemma 3 in [1].

Remark (1) The generalized E.Borel's lemma can be find in [4], which is as a direct corollary of the famous Lojasiewicz Theorem ([4], p79–86). The method used by the author of [1] bases the result on the uniform convergence of the series (1) and (2), which is false, showing by the example above. However, we will see that following function is the one satisfying the conclusion of the theorem:

$$f(x,y) = \frac{\varphi(xy)}{1 - xy}, \quad (x,y) \in R \times R.$$

This is an C^∞ function on $R \times R$. It is easy to obtain that:

$$\frac{\partial^p}{\partial y^p} f(x, 0) = p! x^p, \quad \forall x \in R.$$

As an example, we give the calculation of $p = 1$. For any $x \in R$, there is always an neighborhood U of $0 \in R$, for all $y \in U, |xy| \leq \frac{1}{2}$. So

$$f(x,y) = \frac{1}{1 - xy}, \quad (x,y) \in R \times U.$$

and

$$\frac{\partial}{\partial y}f(x,0)=x.$$

But function $f(x,y)$ can not be gotten from the series (1).

(2) The Theorem 1 and Theorem 2 (Global Whitney's Lemma) in [1] also are the conclusions in [4] (p79–87). The Whitney Lemma and the canonical division theorem of order two are equivalent. In [4] there is a wonderful proof of Whitney Lemma. This proof is based on complex variable, and shows that the conclusions of the two theorems above are also true globally.

References:

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"Whitney 引理的推广及应用" 一文中的错误

曹 义

(贵州大学数学系，贵阳 550025)

摘 要: 本文用一个十分简单的例子说明 [1] 对整体的 Borel 定理的证明是错误的. 为此，还须介绍函数芽和函数芽序列一致收敛的概念，并给出一个判定引理.