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On Poincaré's Remark and a Kind of Nonstandard Measure Defined on $^*(R^n)$ *

L.C. Hsu[1], *SUN Guang-run*[2]

(1. Dept. of Math., Dalian University of Technology, 116024, China;
2. Dept. of Math., Qufu Normal University, Shandong 273165, China)

Abstract: Here concerned is a certain kind of non-standard measure defined on the n-dimensional Euclidean space $^*(R^n)$, which (with $n = 1$) can be used to show that any standard linear point-set or the usual ordered field R of real numbers is of measure zero. The proposition just mentioned is basically consistent with Poincaré's famous remark which renders a deep insight into the paradoxical structural nature of Cantor's continuum consisting precisely of all distinct real numbers.

Key words: Poincaré's remark; Cantor's continuum; enlargement; hyperfinite set; hyperstandard measure.

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1. Introduction

Poincaré's remark is concerned with the nature of Cantor continuum of real numbers (positional points of the line) and also with Cantor's cardinality formula $2^{\aleph_0} = c$ for the continuum of real numbers. It was mentioned in B. Russell's classic book, The Principles of Mathematics ([5], p.347), and it reads as follows: "The continuum thus conceived is nothing but a collection of individuals arranged in a certain order, infinite in number, it is true, but external to each other. This is not the ordinary concept, in which there is supposed to be, between the elements of the continuum, a sort of intimate bond which makes a whole of them, in which the point is not prior to the line, but the line to the point. Of the famous formula $2^{\aleph_0} = c$, the continuum is unity in multiplicity, the multiplicity alone subsists, the unity has disappeared."

As one may see it clearly, the first and third parts of Poincaré's remark agree substantially with Aristotle's viewpoint: "Real numbers (with multiplicity in nature) cannot create a continuum (with unity in nature), since distinct numbers don't touch (connect) each other (or in other words, they are external to each other)." (cf.[1],[5],[6]).

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The second part of remark requires a little more explanation. It is evident that a line (namely, the linear continuum) may have distinct positional points (namely, various point-positions), and the point-position as a concept, can only possibly be assigned with regard to a certain given line. In other words, the point concept is derived from the line. This is why Poincaré remarked that the line is prior to the point, and not conversely.

As having been observed before, Poincaré had already realized that distinct positional points (just corresponding to distinct real numbers in accordance with Cantor's axiom) alone cannot generate a line (continuum), and that the continuum as a unity could only be created with the aid of the so-called "intimate bond" which could make a whole of the points into the line. But, what is the intimate bond?

Intuitively, the intimate bond should be something that is capable of connecting all the positional points into the whole line, and thus creating the continuity nature of the line. As one sees, in the word of nonstandard analysis, any two monads or haloes of hyperreal number are equal or disjoint and the halo has no bound. The haloes revealing aesthetic feeling in images are similar to the "intimate bonds" in Poincaré's mind by which the multiplicity is "softened" into a unity. Giving the rein to one's imagination, Cantor's continuum turned out to be Leibniz's continuum. In nonstandard standpoint, it is well known that the interval [0,1] in R is a discrete positional point set embedded in hyperreal number field ${}^*R$. Thus it is interesting to reckon that the standard interval [0,1] is of some nonstandard measure zero in a still greater universe of discourse, ${}^*R$.

The studies of nonstandard measure theory begin with Robinson's work[4], which continues to use the traditional standard Lebesgue measure ideas. Bernstein and Wattenberg[7] developed the mode of finite thinking and so obtained Lebesque measure on ${}^*[0,1]$ with the aid of counting measure on a hyperfinite subset of ${}^*[0,1]$. Loeb ([8],[9]) showed how convert an internal finitely additive measure space into a standard σ-additive measure space. These constructions of measures and others play important roles in different stages of nonstandard mathematics. All the work has established the representation of standard Lebesgue measure. Using the similar idea as that invented by Dedekind for creating the "continuous" structure of R by means of introducing irrationals into R via Dedekind's cuts of the set of rational numbers, C.G.Huang has just recently succeeded in producing a proof, based on a kind of extended Dedekind section, for the existence of the indivisible continuum of Pythagoras, Democritus, Plato and Galileo (cf.[3]). He has also shown that the set of real numbers is of measure zero in accordance with a kind of measure introduced by himself.

As is mainly inspired by Poincaré's remark, our present paper will expound a new kind of nonstandard measure, so-called hyperstandard measure on ${}^*(R^n)$. In particular it can also be employed in proving the zero-measure property of R. Our basic idea consists of the following essential points. (i) An enlargement implies some hyperfinite represention of each standard set; (ii) It is permissible to use internal (not being *-standard) infinitesimal intervals as covering sets of the sets in question.

Though the reader is assumed to be familiar with basic ideas and terminologies in nonstandard analysis (cf.[2],[4],[6]), some related results will still be given briefly for convenience to quote.

2. Preliminaries

Throughout the paper we work with a superstructure $V(^*R)$ which is κ-saturated, where κ stands for an uncountable number. Suppose that $\mathrm{card}(V(R)) < \kappa$, so that $V(^*R)$ is also an enlargement of the superstructure $V(R)$. This implies the next well-known result.

Theorem 2.1 *For any standard set $A \in V(R)$ there exists a hyperfinite set $F \subseteq {}^*A$ such that F contains all standard elements of ${}^*A$; that is, we have $\{^*a | a \in A\} \subseteq F \subseteq {}^*A$.*

3. A kind of nonstandard measure on $^*(R^n)$

In the classical Lebesgue measure theory on the n-dimensional space R^n, it is allowed to make use of countable infinitely many open intervals $\{I_i\}(i \in N)$ with non-negative volumes as covering sets for any given set E in R, where

$$I_i = E\{x | a_k^{(i)} < x_k^{(i)} < b_k^{(i)},\ k = 1, 2, \cdots, n;\ a_k^{(i)} \leq b_k^{(i)}\},$$

and the volumes of intervals I_i are given by

$$\lambda(I_i) = \prod_{k=1}^{n} (b_k^{(i)} - a_k^{(i)}).$$

The basic principle to be used for computing Lebesgue measures is the so-called axiom of σ-additivity.

In our case for treating a new kind of non-standard measure on $^*(R^n)$, it requires to make use of infinitely many (internal) open intervals $\{I_i\}$ as covering sets of E in $^*(R^n)$, where i ranges over all the natural numbers of $^*N$ (if necessary, by setting $I_n = \emptyset$ for $n > \Omega, \Omega \in {}^*N_\infty$), and moreover, every open interval may have infinitesimal volume.

Definition 3.1 *Given any set $E \subseteq (^*R^n)$, which may be internal or external. Suppose that $\{I_i\}$ is an infinite sequence of pairwise disjoint open intervals of $(^*R^n)$ and that*

$$\bigcup_{i \in {}^*N} I_i \supseteq E.$$

If, in particular, $I_n = \emptyset$ for all $n > \Omega$ with $\Omega \in {}^*N, \{I_i\}$ contains hyperfinitely many intervals. Let the volume of I_i be denoted by

$$\lambda(I_i) = \prod_{k=1}^{n} (b_k^{(i)} - a_k^{(i)}),$$

where $a_k^{(i)}, b_k^{(i)} \in {}^*R, a_k^{(i)} \leq b_k^{(i)}$. Then the number defined by

$$\mu_h^+(E) = \inf_{I_i}\{\mathrm{st}[\sum_{i \in {}^*N} \lambda(I_i)] | E \subseteq \bigcup_{i \in {}^*N} I_i\}$$

is called the outer hyperstandard measure of E, or simply outer h-measure, where we define the mapping

$$\mathrm{st} : {}^*R \to \overline{R} = R \cup \{-\infty, +\infty\}$$

by the following

$$\operatorname{st}(x) = \begin{cases} \operatorname{st}(x), & \text{if } x \in G(0) = \{y \in {}^*R \mid y \sim 0\} \\ +\infty, & \text{if } x \text{ is positive infinite} \\ -\infty, & \text{if } x \text{ is negative infinite} \end{cases}$$

Some properties of the outer h-measure follow from Definition 3.1.

Theorem 3.2 *(i)* $0 \leq \mu_h^+(E) \leq +\infty$ *for all* $E \subseteq {}^*(R^n)$,
(ii) $\mu_h^+(\emptyset) = 0$,
(iii) $\mu_h^+(A) \leq \mu_h^+(B)$ *if* $A \subseteq B \subseteq ({}^*R^n)$.

Definition 3.3 *Let E be a *-bounded set in ${}^*(R^n)$ and let I be any interval in ${}^*(R^n)$ such that $I \supseteq E$. Then*

$$\mu_h^-(E) = \operatorname{st}(\lambda(I)) - \mu_h^+(I \backslash E)$$

is called the inner hyperstandard measure of E, or simply inner h-measure.

Definition 3.4 *A *-bounded set $E \subseteq {}^*(R^n)$ is said to be h-measurable if $\mu_h^+(E) = \mu_h^-(E)$. In this case we denote*

$$\mu_h(E) = \mu_h^+(E) = \mu_h^-(E),$$

and call it the hyperstandard measure of E, or simply h-measure.

For a *-unbounded set $E \subseteq {}^*(R^n)$, it is h-measurable if $E \cap I$ is *-bounded and h-measurable, where I is any open interval in ${}^*(R^n)$.

Remark 3.5 It is obvious that the present defintions in ${}^*(R^n)$ are consistent with the standard notions of Lebesgue measure in R^n.

Consequence 3.6 Definition 3.3 implies

$$\mu_h^-(I \backslash E) = \operatorname{st}[\lambda(I)] - \mu_h^+(E).$$

If E is h-measurable, then $\mu_h^-(E) = \mu_h^+(E)$. Thus we have

$$\mu_h^-(I \backslash E) = \operatorname{st}[\lambda(I)] - \mu_h^-(E) = \mu_h^+(I \backslash E).$$

This shows that the set $I \backslash E$ is also h-measureable, and

$$\mu_h(E) + \mu_h(I \backslash E) = \operatorname{st}[\lambda(I)].$$

4. A proof that R is of h-measure zero

In particular, for the linear sets in ${}^*R$, it is easy to show that $\mu_h({}^*[0,1]) = \operatorname{st}[\lambda({}^*[0,1])] = 1$ and $\mu_h^+({}^*R) = +\infty$. But we are able to prove $\mu_h(R) = 0$. For this purpose, let us first establish the following

Proposition 4.1 *Let $F \subseteq {}^*R$ be any hyperfinite set. Then we have $\mu_h^+(F) = 0$.*

Proof Let F be denoted by

$$F = \{x_1, x_2, \cdots, x_\Omega\},$$

where the elements are arranged in its usual ordering $^* \leq$. We use $|F|$ to denote the internal cardinality Ω of F.

Assume that $\Omega \in {}^*N_\infty$ (it is trivial if Ω is finite). Let

$$\min_{2 \leq j \leq \Omega} (x_j - x_{j-1}) = d,$$

and suppose that ε is any positive infinitesimal in $^*R$ such that $\varepsilon < d$. Let I_i be defined by

$$I_i = (x_i - \frac{\varepsilon}{\Omega}, x_i + \frac{\varepsilon}{\Omega}), \quad i = 1, 2, \cdots, \Omega.$$

Evidently, $\{I_i\}$ is a pairwise disjoint family of sets and

$$\bigcup_{1 \leq i \leq \Omega} I_i \supseteq F.$$

Consequently we have

$$0 \leq \mu_h^+(F) \leq \mathrm{st}(\sum_{i=1}^{\Omega} \lambda(I_i)) = \mathrm{st}(\Omega \frac{2\varepsilon}{\Omega}) = \mathrm{st}(2\varepsilon) = 0.$$

This proves that $\mu_h^+(F) = 0$. □

Proposition 4.2 *We have* $\mu_h(R) = 0$.

Proof It suffices to consider the linear set $[0,1] \subset R$, and show that $\mu_h([0,1]) = 0$.

By Theorem 2.1, there exists a hyperfinite set F such that

$$[0,1] \subseteq F \subseteq {}^*[0,1].$$

It is immediate from Theorem 3.2 and Proposition 4.1 that $\mu_h^+([0,1]) \leq \mu_h^+(F) = 0$, and so that $\mu_h^+([0,1]) = 0$. Clearly, from Definition 3.1 we have $\mu_h^+([0,1]) \geq \mu_h^-([0,1])$, and consequently, $\mu_h^-([0,1]) = 0$. Hence we may infer that $\mu_h([0,1]) = 0$. □

Remark 4.3 All what we have expounded in this paper is the basic proposition that R is a discrete structure embedded in $^*R$, so that the h-measure of R is zero. The proposition, to a certain extent, is similar to the situation for the set Q of rational numbers, which is an obvious discrete structure embedded in R, with Lebesgue measure zero.

But why should we all believe in the Lebesgue measure theory so that $[0,1] \backslash Q$ is of measure 1? The simple reason may be that, in the classical analysis working with R, the universe R of discoure is too narrow to squeeze Poincaré's so-called "intimate bond" of real points in it. So one can only do his best by making use of at most countably infinitely many intervals of positive real lengths (volumes) (not being infinitesimal) as covering sets. In fact, the analytic tools for measuring sets in the Lebesgue case cannot be further refined anyway without appealing to the nonstanded infinitesimal analysis. The present investigation once again shows that the structure of $^*R(^*(R^n))$ filled with "ideal elements" or intimate bond is far more rich and ingenious than that of $R(R^n)$, so that much more delicate measuring technics (apart from the powerful Loeb measure theory on an internal set X, etc.) could still be constructed for the sets of $^*(R^n)$.

Note added The main result of this paper was reported at the International Symposium on Analysis, Combinatorics and Computing (Dalian) on August 5, 2000.

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关于 Poincaré 氏注记和定义在 $^*(R^n)$ 上的一种非标准测度

徐利治[1]，孙广润[2]

(1. 大连理工大学应用数学系，辽宁 116024; 2. 曲阜师范大学数学系，山东 273165)

摘 要: 受到 Poincaré 氏评注的启发，本文引入了一种新的非标准测度，简称为 μ_h 测度. 文中应用非标准分析方法证明了标准实数集 R 在 $^*R$ 上的 μ_h 测度为零. 文末还指出了 μ_h 测度与 Lebesgue 测度具有相异本质的原由.