Journal of Mathematical Research & Exposition
Vol.21, No.2, 165-170, May, 2001

C^∞ Compactness for a Class of Riemannian Manifolds with Parallel Ricci Curvature *

XU Sen-lin, *MEI Jia-qiang*
(Dept. of Math., Univ. of Sci. and Tech. of China, Hefei 230026, China)

Abstract: In this paper we prove that the set of Riemannian manifolds with parallel Ricci curvature, lower bounds for sectional curvature and injectivity radius and a upper bound for volume is c^∞ compact in Gromov-Hausdroff topology. As an application we also prove a pinching result which states that a Ricci flat manifold is flat under certain conditions.

Key words: sectional curvature; Ricci curvature; injectivity radius; diameter; volume; Jacobi field.

Classification: AMS(1991) 53C20/CLC O186

Document code: A **Article ID:** 1000-341X(2001)02-0165-06

1. Introduction

In this paper we consider $n-$dimensional closed Riemannian manifolds. Suppose M is such a manifold. The sectional curvature, Ricci curvature, injectivity radius, diameter and volume of M are denoted by $k(M)$,$\mathrm{Ric}(M), i(M), d(M)$and $V(M)$ respectively. The Riemannian curvature tensor is denoted by Rm. The well-known convergence theorem says that the set of manifolds satisfying the bounds

$$|k(M)| \le K, V(M) \ge V, d(M) \le D$$

is $c^{1,\alpha}$ compact. The c^k version of this Cheeger-Gromov compactness theorem then states that the space of manifolds satisfying the bounds

$$|\nabla^j \mathrm{Rm}| \le \Lambda, j \le k, V(M) \ge V, d(M) \le D$$

is $c^{k+1,\alpha}$ compact. Where $|\nabla^j Rm|$ is the point-wise norm of the $j-$th covariant derivative of Rm. We will study c^∞ compactness for certain manifolds. We say that a set of manifolds

*Received date: 1998-05-29
Foundation item: Supported by National Natural Science Foundation of China (19971081)
Biography: XU Sen-lin (1941-), male, born in Wu county, Jiangsu province. Currently professor at USTC.
E-mail: xusl@nstc.edu.cn

is c^∞ compact, if given a sequence of M_i in this set, there are a subsequence which is also denoted by $\{M_i\}$, a manifold M(belonging to this set), a sequence of diffeomorphisms $f_i : M \to M_i$, such that $\{f_i^* g_i\}$ converges to g in c^k topology for every k. Where g_i is the Riemannian mertic on M_i,g is a Riemannian metic on M.

Let $\Gamma(n, \lambda, i_0, V)$ be the class of $n-$dimensional($n \geq 3$) closed Riemannian manifolds satisyfing

$$\nabla \mathrm{Ric} = 0, \tag{1}$$

$$k(M) \geq \lambda, \tag{2}$$

$$i(M) \geq i_0, \tag{3}$$

$$V(M) \leq V. \tag{4}$$

Then we have

Theorem 1 *$\Gamma(n, \lambda, i_0, V)$ is c^∞ compact.*

As an application of this theorem, we have

Theorem 2 *For every n and a positive real number i_0 and a real number λ, there exists a poisitive real number $\varepsilon(n, i_0)$ such that the $n-$dimensional closed Ricci flat Riemannian manifolds satisfying*

$$|\nabla \mathrm{Rm}| \leq \varepsilon(n, \lambda, i_0), k(M) \geq \lambda, d(M) = 1, i(M) \geq i_0$$

must be flat.

This result may be compared with rigidity theorems in [6],[7].

2. Preliminaries

Suppose M is a closed manifold of dimension n. $\{\omega_1, \cdots, \omega_n\}$ is the dual orthonormal cotangent frame. We have the structure equations:

$$d\omega_i = -\sum_j \omega_{ij} \wedge \omega_j,$$

$$d\omega_{ij} = -\sum_k \omega_{ik} \wedge \omega_{kj} + \Omega_{ij},$$

where $\Omega_{ij} = \frac{1}{2}\sum_{kl} R_{ijkl}\omega_k \wedge \omega_l$, R_{ijkl} is the component of Riemannian curvature tensor.the indexes range from 1 to n. The component of Ricci curvature tensor R_{ij} is defined by $\sum_l R_{lilj}$.

The covariant derivative is defined by

$$\sum_k R_{ij,k}\omega_k = dR_{ij} - \sum_m R_{mj}\omega_{mi} - \sum_m R_{im}\omega_{mj},$$

$\nabla \mathrm{Ric} = 0$ means $R_{ij,k} = 0$ for all i, j, k.

The covariant derivative of curvature tensor is defined by

$$\sum_m R_{ijkl,m}\omega_m = dR_{ijkl} - \sum_n R_{njkl}\omega_{nm} - \sum_n R_{inkl}\omega_{nm} - \sum_n R_{ijnl}\omega_{nk} - \sum_n R_{ijkn}\omega_{nm}.$$

Similaryly, one can define the higher order covariant derivative. We have the Ricci identity

$$R_{i_1i_2i_3i_4,i_5i_6,\cdots,i_rmn} - R_{i_1i_2i_3i_4,i_5i_6,\cdots,i_rnm} = \sum_{rj} R_{i_1,\cdots,i_{j-1}ri_{j+1},\cdots,i_r} R_{ri_jmn}. \tag{5}$$

If $\nabla\mathrm{Ric} = 0$, we have(cf.[8])

$$\begin{aligned}\sum_m R_{ijklmm} = 2\sum_{n,m}(R_{hjkl}R_{hilm} + R_{ihkm}R_{hjlm} + R_{ijhm}R_{hklm}) + \\ \sum_h (R_{ijkh}R_{hl} - R_{ijlm}R_{hk}).\end{aligned} \tag{6}$$

More generally,we have($r \geq 5$) :

$$\begin{aligned}\sum_m R_{i_1i_2,\cdots,i_rmm} = \sum_m (R_{i_1i_2,\cdots,i_{r-1}i_rm} - R_{i_1i_2,\cdots,i_{r-1}mi_r})_m + \\ \sum_m (R_{i_1,\cdots,i_{r-1}mi_rm} - R_{i_1,\cdots,i_{r-1}mmi_r}) + (\sum_m R_{i_1,\cdots,i_{r-1}mm})_{i_r}.\end{aligned} \tag{7}$$

These equations, combined with the following Sobolev inequality, will be used to estimated the point-wise norm $|\nabla^j \mathrm{Rm}|$, where $|\nabla^j\mathrm{Rm}|^2 = \sum_{i_1,\cdots,i_{j+4}} R^2_{i_1,\cdots,i_{j+4}}$. The Sobolev inequality(cf.[9],Appendix 2)states that, if $\mathrm{Ric}(M)d^2(M) \geq \mu$, then there exists a positive number $\gamma = \gamma(n, d(M), \mu)$, such that, for $n \geq 3$,

$$||f||_{\frac{2n}{n-2}} \leq V(M)^{-\frac{1}{n}}[\gamma||df||_2 + ||f||_2], \forall f \in W^{1,2}(M). \tag{8}$$

3. Proofs of theorem 1 and theorem 2

First let's introduce some lemmas.

Lemma 1 *If $M \in \Gamma(n, \lambda, i_0, V)$, then $k(M) \leq (n-1)\pi^2/i_0^2 - (n-2)\lambda$.*

Proof Since $k(M) \geq \lambda$, we only need to prove that $\mathrm{Ric}(M) \leq (n-1)\pi^2/i_0^2$. This follows from a Jacobi field argument. Choose orthonormal frame $\{e_1, e_2, \cdots, e_n\}$ at $p \in M$. We will prove that $\mathrm{Ric}(e_1, e_1) \leq (n-1)\pi^2/i_0^2$. Let $\gamma(t) = \exp_p(te_1)$ be the geodesic which is minimal when $t \in [0, i_0]$. Put $U_i(t) = \sin(\frac{\pi}{i_0}t)e_i(t)$. Where $e_i(t)$ is the parallel translation of e_i along γ, $2 \leq i \leq n$. Let $l_i(s)$ be the length of the curve $\sigma_{i,s}$, where $\sigma_{i,s} = \exp_{\gamma(t)} sU_i(t), t \in [0, i_0]$. Since $\gamma|_{[0,i_0]}$ is minimal, $l_i''(0) \geq 0$. So by the second variation formula of arc length, we have

$$\begin{aligned}0 \leq \sum_{i=2}^n l_i''(0) &= \sum_{i=2}^n \int_0^{i_0} \{|\dot{U}_i(t)|^2 - \langle R_{\dot\gamma U_i}\dot\gamma, U_i\rangle\}\mathrm{d}t \\ &= \sum_{i=2}^n \int_0^{i_0} \{\frac{\pi^2}{i_0^2}cos^2\frac{\pi}{i_0}t_0 - sin^2\frac{\pi}{i_0}t_0\mathrm{Ric}(\dot\gamma, \dot\gamma)\}\mathrm{d}t \\ &= \frac{1}{2}[(n-1)\frac{\pi^2}{i_0^2} - \mathrm{Ric}(e_1, e_1)].\end{aligned}$$

Where we have used the fact that $\mathrm{Ric}(\dot\gamma,\dot\gamma)\equiv \mathrm{Ric}(e_1,e_1)$, which is guaranteed by the condition $\nabla Ric=0$. Since e_1 is a arbitrary unit vector, we are done.

Lemma 2 *If $M\in\Gamma(n,\lambda,i_0,V)$,then $v(M)\geq v(n,i_0),d(M)\leq D(n,i_0,V)$,where $v(n,i_0)$ is a constant depending only on n,i_0. $D(n,i_0,V)$ has similar meaning.*

Proof $\forall p\in M, V(M)\geq \mathrm{vol}B(p,i_0)$, where $B(p,i_0)$ is the geodesic ball with center p and radius i_0. It follows from [10]that $\mathrm{vol}B(p,i_0)\geq v(n,i_0)$. So the first assertion is settled. The second assertion follows from a simple packing argment. Suppose $2i_0\leq d(M)\leq 2(k+1)i_0$,$k$ is a nongetive integer. Choose $x,y\in M$ such that the minimal geodesic connecting x and y has length $d(M)$. There are points$\{x_1,x_2,\cdots,x_{k+1}\}$ on this geodesic such that the distance between x_i and x_j is $2i_0$ for $i\neq j$. So $B(x_i,i_0)\cap B(x_j,i_0)=\emptyset$ for $i\neq j$. then we have

$$V\geq V(M)\geq\sum_{i=1}^{k+1}\mathrm{vol}B(x_i,i_0)\geq(k+1)v(n,i_0).$$

This yields $d(M)\leq 2(k+1)i_0\leq 2i_0V/v(n,i_0)$.

Now it follows from lemma 1 and 2 that $\Gamma(n,\lambda,i_0,V)$ is $c^{1,\alpha}$ compact. Also, $\Gamma(n,\lambda,i_0,V)$ contians only finitely many diffeomorphism types of manifolds. We are now going to prove that $\Gamma(n,\lambda,i_0,V)$ is c^∞ compact.

Proof of Theorem 1 We only need to prove that $|\nabla^j\mathrm{Rm}|\leq c_j$ for every j. Where c_j is a constant depend only on n,λ,i_0,V and j. For $j=0$, lemma 1 gives us the bound. Applying Stoke's formula on $\frac{1}{2}\triangle|\mathrm{Rm}|^2=-|\nabla\mathrm{Rm}|^2+\sum_{ijklm}R_{ijkl}R_{ijklmm}$ we have $\int_M|\nabla\mathrm{Rm}|^2\leq d_1$. Where $\triangle=-tr\nabla^2$ is the Lapacian, d_1 and the following d_k are constants. Suppose we already have $|\nabla^j\mathrm{Rm}|\leq c_j$ for $j\leq k-1$, $\int_M|\nabla^k\mathrm{Rm}|^2\leq d_k$. Then by (5),(6)and(7), we have

$$\frac{1}{2}\triangle|\nabla^k\mathrm{Rm}|^2\leq -|\nabla^{k+1}\mathrm{Rm}|^2+a|\nabla^k\mathrm{Rm}|^2+b|\nabla^k\mathrm{Rm}|, \tag{9}$$

a,b are positive constants depend on $c_j, j\leq k-1$. Now recall Kato's inequality $|d|\nabla^k\mathrm{Rm}||^2\leq|\nabla^{k+1}\mathrm{Rm}|^2$, we have

$$\begin{aligned}|\nabla^k\mathrm{Rm}|\triangle|\nabla^k\mathrm{Rm}| &\leq |d|\nabla^k\mathrm{Rm}|^2-|\nabla^{k+1}\mathrm{Rm}|^2+a|\nabla^k\mathrm{Rm}|^2+b|\nabla^k\mathrm{Rm}|\\ &\leq a|\nabla^k\mathrm{Rm}|^2+b|\nabla^k\mathrm{Rm}|.\end{aligned}$$

We can rewrite this inequality as

$$\triangle(|\nabla^k\mathrm{Rm}|+b/a)\leq a(|\nabla\mathrm{Rm}|+b/a). \tag{10}$$

Put $f=|\nabla^k\mathrm{Rm}|+b/a$, by Kato's inequality, (9)and $\int_M|\nabla^k\mathrm{Rm}|^2\leq d_k$, we know that $\int_M|\nabla^{k+1}\mathrm{Rm}|^2\leq d_{k+1}, f\in W^{1,2}(M)$. With (10)and the Sobolev inequality (8), one can deduce the bound $|\nabla^k\mathrm{Rm}|+b/a=f\leq c_k$ by standard Moser's iteration(cf.[9]).

So inductively we get the bounds $|\nabla^j\mathrm{Rm}|\leq c_j$ for all j. Thus the c^k version of Cheeger-Gromov compactness theorem yields that $\Gamma(n,\lambda,i_0,V)$ is c^∞ compact.

Remark 1 If we consider locally symmetric manifolds, namely, condition(1) is replaced by "$\nabla Rm = 0$",then condition(2) can be replaced by "scalar curvature of $M \geq \lambda$".This is because the method of lemma 1 now gives the bound $k(M) \leq \pi^2/i_0^2$. So actully $k(M)$ is bounded. In particular, for $n = 3$, $\nabla Ric = 0$ implies $\nabla Rm = 0$ since three dimensional manifolds have vanishing Weyl conformal curvature tensor. Thus the condition may be weakened.

Remark 2 For $n = 4$, condition can replaced as above. This is because $\nabla \mathrm{Ric} = 0$ implies $\nabla \mathrm{Rm} = 0$ or M is an Einstein manifold for $n = 4$ (cf.[11]). While for four dimensional Einstein manifolds, there is already a c^∞ compactness theorem in[12].

Now we go to prove Theorem 2, which gives an application of Theorem 1.

Proof of Theorem 2 Suppose on the contrary that there is a sequence of $n-$ dimensional non-flat manifolds $\{M_j\}$ satisfying $\mathrm{Ric} \equiv 0, |\nabla \mathrm{Rm}| \leq 1/j, k(M) \geq \lambda, d(M) = 1, i(M_j) \geq i_0$. The Bishop-Gromov volume comparision theorem yields that $V(M) \leq \omega_n$, where ω_n is the volume of the unit ball of R^n with standard metric. By Theorem 1, there are a $n-$dimensional manifold M and a subsequence of $\{M_j\}$ c^∞ converging to M. Without loss of generality,we can assume that $\{M_j\}$ itself c^∞ converges to M. Namely,there is a sequence of diffeomorphism $f_i : M \to M_i$ such that $\{f_i^* g_i\} c^\infty$ converges to g. Where g_i is the Riemannian metric on M_i, g is a Riemannian metric on M. Now M is Ricci flat and locally symmetric, so M is flat(cf.[13]). By passing to finite covering we can assume that M is the $n-$dimensional torus T^n. Now the Riemannian manifold $(T^n, f_i^* g_i)$ is Ricci flat, so it is actually flat(cf[14],[15]). This yields that (M_i, g_i) is flat, which contradicts to our assumption.

Remark 3 We feel that the condition "$k(M) \geq \lambda$" in Theorem 2 may be removed. At least for $n = 4$ this is true by remark 2.

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Ricci 曲率平行的一类 Riemann 流形的 C^∞ 紧性

徐 森 林， 梅 加 强
(中国科技大学数学系，合肥 230026)

摘 要: 本文证明，在 Gromov-Hausdorff 拓扑下，Ricci 曲率平行，截面曲率和单一半径有下界，体积有上界的 Riemann 流形的集合是 c^∞ 紧的. 作为应用，我们证明一个 pinching 结果，即在某些条件下， Ricci 平坦的流形必定平坦.