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Approximation Properties of a Class of Extremal Polynomials over Smooth Jordan Curves *

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Abstract: There have been many elegant results discussing the approximation properties of the Bieberbach polynomials. However, very few papers investigated the approximation properties of the extremal polynomials over Jordan curves. In the present paper, some results on a class of extremal polynomials over $C^{1+\alpha}$ smooth Jordan curves are obtained.

Key words: smooth Jordan curve; extremal polynomial; approximation property.

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1. Introduction

Let D be a simple connected Jordan domain with $0 \in D$, $\Gamma = \partial D$, let $\phi(z)$ be the conformal mapping of D onto the unit circle disk $U = \{w : |w| < 1\}$ with $\phi(0) = 0$, $\phi'(0) > 0$, and $\psi(w)$ the inverse mapping of ϕ. We denote by $E^p(D)$ the space of all functions $f(z)$ analytic in D and satisfy

$$\sup_{0<r<1} \int_0^{2\pi} |f \circ \psi(re^{i\theta})|^p |\psi'(re^{i\theta})| \mathrm{d}\theta < +\infty.$$

It is well-known that when Γ satisfies the Smirnov condition, that is,

$$\exp[\frac{1}{2\pi} \int_0^{2\pi} \log|\psi'(e^{i\theta})| \frac{e^{i\theta}+w}{e^{i\theta}-w} \mathrm{d}\theta] = \psi'(w), \quad |w| < 1,$$

then the polynomials are dense in $E^p(D)$ for any $p > 0$ (see [5]).

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Denote by $M_p(D)$ the space of all functions $f(z)$ analytic in D and $f'(z) \in E^p(D)$ with the normalization condition $f(0) = 0$, $f'(0) = 1$. We know that there is an extremal function $f_p(z)$ minimizing the integral

$$[\int_\Gamma |f'(z)|^p |\mathrm{d}z|]^{1/p} = \|f'\|_p, \tag{1.1}$$

also, the extremal function $f_p(z)$ is the following function

$$f_p(z) = \int_0^z [\phi'(\xi)/\phi'(0)]^{1/p} \mathrm{d}\xi, \quad z \in D,$$

Furthermore, we have

$$\min_{f \in M_p} \|f'\|_p^p = \|f_p'\|_p^p = \int_\Gamma |\phi'(z)/\phi'(0)||\mathrm{d}z| = 2\pi\psi'(0).$$

Let Π_n be the class of all polynomials $P_n(z)$ of degree $\leq n$ with the normalization condition $P_n(0) = 0$, $P_n'(0) = 1$. We know that there is exactly one polynomial $P_{n,p}(z) \in \Pi_n$ minimizing the integral (1.1), and that the extremal polynomial $P_{n,p}(z)$ converges to $f_p(z)$ uniformly on any compact subsets of D when Γ satisfies the Smirnov condition (see [5]).

The present paper gives an estimate to

$$\|f_p - P_{n,p}\|_c := \max_{z \in \bar{D}} |f_p(z) - P_{n,p}(z)|$$

for any smooth Jordan curve $\Gamma \in C^{1+\alpha}$, $0 < \alpha < 1$, and $p = 2$.

From [8], when $\Gamma \in C^{1+\alpha}, 0 < \alpha < 1$, $\psi'(w)$ and $\phi'(z)$ have continuous extensions to $\bar{U}$ and $\bar{D}$, and there are constants $A_1 > 0$, $A_2 > 0$ such that

$$\begin{aligned} A_1 \leq |\psi'(w)| \leq A_2, \quad |w| \leq 1, \\ A_1 \leq |\phi'(z)| \leq A_2, \quad z \in \bar{D}, \end{aligned} \tag{1.2}$$

as well as

$$\begin{aligned} \psi'(w) \in \mathrm{Lip}\alpha, \quad |w| \leq 1, \\ \phi'(z) \in \mathrm{Lip}\alpha, \quad z \in \bar{D}. \end{aligned} \tag{1.3}$$

The main result of this paper is the following

Theorem 1 *Let D be bounded by a smooth curve $\Gamma \in C^{1+\alpha}, 0 < \alpha < 1$. Then there exists a constant $C > 0$ such that*

$$\max_{z \in \bar{D}} |f_2(z) - P_{n,2}(z)| \leq Cn^{-\alpha} \log(n+1) \tag{1.4}$$

2. Preliminaries

To prove (1.4), we first obtain an estimate of $[\int_\Gamma |f_2'(z) - P_n'(z)|^2 |dz|]^{1/2}$ for an arbitrary polynomial $P_n(z) \in \Pi_n$. For this purpose, we will work with the modulus of continuity of f_2' to $\bar{D}$.

Lemma 2.1 *Let $\Gamma \in C^{1+\alpha}, 0 < \alpha < 1$. Then $f_2'(z)$ has a continuous extension to $\bar{D}$ such that its modulus of continuity satisfies*

$$\omega(f_2', \delta) = O(\delta).$$

Proof It is obvious that $f_2'(z) = [\phi'(z)/\phi'(0)]^{1/2}, \phi'(z) \neq 0, z \in D$. Since $\Gamma \in C^{1+\alpha}$, by Theorem 3.5 in [7], it follows that $\phi'(z)$ has a continuous extension to$\bar{D}$ and $\phi'(z) \neq 0, z \in \bar{D}$, and so that $f_2'(z)$ has a continuous extension to $\bar{D}$.

From (1.2) and (1.3), it can be deduced from [4] that

$$\phi''(z) = O(\mathrm{dist}(z, \Gamma)^{\alpha-1}), z \in D,$$

$$f_2''(z) = \frac{1}{2\phi'(z)}[\frac{\phi'(z)}{\phi'(0)}]^{1/2} \times \phi''(z) = O(dist(z, \Gamma)^{\alpha-1}), z \in D.$$

Thus, it follows from [4] that

$$f_2'(z) \in \mathrm{Lip}\alpha, z \in \bar{D}. \qquad \square$$

We come to construct polynomials $T_n(z) \in \Pi_n$ such that $T_n'(z)$ can approximate $f_2'(z)$ in the norm of $E^2(D)$. Since this result has its own independent value, we write it as the following

Theorem 2 *If $\Gamma \in C^{1+\alpha}, 0 < \alpha < 1$, then there exists a polynomial $T_n(z) \in \Pi_n$ such that*

$$\max_{z \in \bar{D}} |f_2'(z) - T_n'(z)| = O(n^{-\alpha}), \tag{2.1}$$

and $\|f_2' - T_n'\|_2 = O(n^{-\alpha})$.

Proof It is sufficient to prove (2.1) only. By the known result in [2], there exists a polynomial sequence $Q_n(z)$ with $\deg Q_n \le n$ such that $\max_{z \in \bar{D}} |f_2(z) - Q_n(z)| = O(n^{-1-\alpha})$, and

$$\max_{z \in \bar{D}} |f_2'(z) - Q_n'(z)| = O(n^{-\alpha}). \tag{2.2}$$

Setting $T_n(z) = Q_n(z) + (1 - Q_n'(0))z - Q_n(0)$. We have

$$\max_{z \in \bar{D}} |f_2'(z) - T_n'(z)| \le \max_{z \in \bar{D}} |f_2'(z) - Q_n'(z)| + |1 - Q_n'(0)|.$$

By (2.2), we get

$$\max_{z \in \bar{D}} |f_2'(z) - T_n'(z)| = O(n^{-\alpha}). \qquad \square$$

Finally, we establish an inequality similar to the Andrievskii inequality ([1],[3]) for polynomials.

Lemma 2.2 *Let D be a domain bounded by a quasiconformal Jordan curve Γ, and $0 \in D$. Then for every polynomial $P_n(z)$ of degree $n \geq 2$ with $P_n(0) = 0$, we have*

$$\max_{z\in\bar{D}} |P_n(z)| = O([\int_\Gamma |P_n'(z)|^2 |\mathrm{d}z|]^{1/2} \log n).$$

Proof From [6], we know that there exist constants $\beta \in (0,1]$ and $B > 0$ such that

$$|\psi(w_1) - \psi(w_2)| \leq B|w_1 - w_2|^\beta, \quad w_1, w_2 \in \bar{U}.$$

Let $f(z)$ be analytic in D with $f(0) = 0$,

$$(\int_\Gamma |f'(z)|^2 |\mathrm{d}z|)^{1/2} < +\infty.$$

For $w \in U$, $|w| \leq r < \rho < 1$, we have

$$f'(\psi(w))\psi'(w) = \frac{1}{2\pi i}\int_{|\xi|=\rho} \frac{f'(\psi(\xi))\psi'(\xi)}{\xi - w}\mathrm{d}\xi.$$

By integration on the segment $[0, w]$, we have

$$f(\psi(w)) = \frac{1}{2\pi i}\int_{|\xi|=\rho} f'(\psi(\xi))\psi'(\xi)\mathrm{d}\xi \int_0^w \frac{\mathrm{d}\eta}{\xi - \eta}. \tag{2.3}$$

Applying Hölder inequality to (2.3) we get

$$\begin{aligned} |f(\psi(w))| &\leq \frac{1}{2\pi}[\int_{\Gamma_\rho} |f'(z)|^2 |\mathrm{d}z|]^{1/2}[\int_{|\xi|=\rho} |\psi'(\xi)|(\int_0^r (\rho - t)^{-1}\mathrm{d}t)^2 |\mathrm{d}\xi|]^{1/2} \\ &\leq \frac{1}{2\pi} l^{1/2}(\Gamma)(\int_\Gamma |f'(z)|^2 |\mathrm{d}z|)^{1/2} \log\frac{1}{1-r}, \end{aligned} \tag{2.4}$$

where $\Gamma_\rho = \{z = \psi(w) : |w| = \rho\}$ is the inner level curve of Γ, and $l(\Gamma)$ is the length of Γ.

By (2.4), taking $r = 1 - n^{-2/\beta}$, $n \geq 2$, we obtain

$$|f(z)| \leq \frac{1}{\beta\pi} l^{1/2}(\Gamma)[\int_\Gamma |f'(z)|^2 |\mathrm{d}z|]^{1/2} \log n,$$

where $z = \psi(w)$ with $|w| \leq 1 - n^{-2/\beta}$.

Now assume that $f(z)$ is a polynomial of degree $n \geq 2$. Bernstein's lemma (see [9]) states that for any $R > 1$,

$$|f(z)| \leq \frac{1}{\beta\pi} l^{1/2}(\Gamma)[\int_\Gamma |f'(z)|^2 |\mathrm{d}z|]^{1/2} R^n \log n \tag{2.5}$$

for z in the interior of $\Gamma_{r_n,R}$, where $\Gamma_{r_n,R}$ denotes the outer level curve of Γ_{r_n} with $r_n = 1 - n^{-\beta/2}$. It follows from Lemma 1 in [3] that there exists $M > 0$ such that D is contained in the interior of Γ_{r_n,R_n} with $R_n = 1 + M/n$. Hence from (2.5) we deduced that

$$\max_{z\in\bar{D}} |f(z)| = O([\int_\Gamma |f'(z)|^2 |\mathrm{d}z|]^{1/2} \log n). \qquad \square$$

3. The Proof of Theorem 1

Note that for any $p_n(z) \in \Pi_n$,

$$\begin{aligned}
&\int_\Gamma \overline{[\phi'(z)/\phi'(0)]^{1/2}}(P_n'(z) - [\phi'(z)/\phi'(0)]^{1/2})|\mathrm{d}z| \\
&= \int_{|w|=1} \frac{1}{[\phi'(0)]^{1/2}} \frac{|\psi'(w)|}{[\psi'(w)]^{1/2}}(P_n'(\psi(w)) - [\psi'(0)/\psi'(w)]^{1/2})|\mathrm{d}w| \\
&= \frac{1}{i}\int_{|w|=1}[(\frac{|\psi'(w)|}{\psi'(0)})^{1/2}P_n'(\psi(w)) - 1]\frac{\mathrm{d}w}{w} \\
&= 2\pi[P_n'(\psi(0)) - 1] = 0,
\end{aligned}$$

we conclude immediately that for any $P_n(z) \in \Pi_n$,

$$\int_\Gamma |f_2'(z) - P_n'(z)|^2|\mathrm{d}z| = \int_\Gamma |P_n'(z)|^2|\mathrm{d}z| - \int_\Gamma |f_2'(z)|^2|\mathrm{d}z|.$$

Therefore by the extremal property of $P_{n,2}(z)$, it follows from Theorem 2 that

$$[\int_\Gamma |f_2'(z) - P_{n,2}'(z)|^2|\mathrm{d}z|]^{1/2} = O(n^{-\alpha}). \tag{3.1}$$

Applying Minkowskii's inequality to (3.1), we obtain that for n with $2^k \le n < 2^{k+1}$

$$[\int_\Gamma |P_{2^{k+1},2}'(z) - P_{n,2}'(z)|^2|\mathrm{d}z|]^{1/2} = O(n^{-\alpha}). \tag{3.2}$$

On the other hand, we know by the Cauchy's integral formula that $P_{n,2}'(z)$ converges uniformly to $f_2'(z)$ on every compact subsets of D. Consequently, for $z \in \bar{D}$, we have

$$f_2(z) - P_{n,2}(z) = P_{2^{k+1},2}(z) - P_{n,2}(z) + \sum_{j=k+1}^{\infty}[P_{2^{j+1},2}(z) - P_{2^j,2}(z)].$$

Then by Lemma 2.2 and (3.2),

$$\max_{z\in\bar{D}}|f_2(z) - P_{n,2}(z)| = O(\frac{\log n}{n^\alpha}), \quad n = 2, 3, \cdots. \tag{3.3}$$

Evidently, (3.3) implies that there exists a constant $C > 0$ such that

$$\max_{z\in\bar{D}}|f_2(z) - P_{n,2}(z)| \le Cn^{-\alpha}\log(n+1)$$

for any nonnegative integer n.

References:

[1] ANDRIEVSKII V V. *Convergence of Bieberbach polynomials in domains with quasiconformal boundary* [J]. Ukrain. Math. J., 1983, **35**: 233–236.

[2] DZYADYK V K. *Introduction to the Theory of Uniform Approximation of Functions by Polynomials (Russian)* [M]. Izdatel'stvo "Nauka", Moskow 1977.

[3] GAIER D. *On a polynomial lemma of Andrievskii* [J]. Arch. Math., 1987, **49**: 119–123.

[4] GEHRING F W. and MATIO O. *Quasidisks and Hardy-Littlewood property* [J]. Complex variables Theory Appl., 1983, **2**: 67–68.

[5] KELDYCH M and LAVRENTIEFF M. *Sur la representation conforme des domaines limites par des courbes retifiables* [J]. Ann. Ecole. Norm., 1937, **137**(54): 1–38.

[6] LEHTO O and VIRTANEN K I. *Quasiconformal mappings in the plane* [M]. Berlin Heildelberg, New York, 1973.

[7] POMMERENKE C H. *Boundary behavior of conformal maps* [M]. Springer, Berlin, 1992.

[8] VEKYA I N. *Generalized Analytic Functions (Russian)* [M]. Moscow Fizmatgiz, 1959.

[9] WALSH J L. *Interpolation and Approximation by Rational Functions in the Complex Domain* [M]. Providence, R. I., 1969.

光滑 Jordan 曲线上一类极值多项式的逼近性质

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摘 要: 关于 Bieberbach 多项式的逼近性质已有许多精彩结果，然而 Jordan 曲线上极值多项式的逼近性质却很少被考察．本文得到了 $C^{1+\alpha}$ 光滑 Jordan 曲线上一类极值多项式的一些逼近结果．