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Bilocal Derivations of Reflexive Algebras *

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Abstract: Let $\mathcal{A}$ be a reflexive algebra in reflexive Banach space X such that both $O_+ \neq O$ and $X_- \neq X$ in Lat$\mathcal{A}$, then the set of all derivations of $\mathcal{A}$ into $B(X)$ is topologically algebraically bireflexive.

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1. Introduction

Let X be a Banach space over the complex field and $B(X)$ denote the algebra of all bounded linear operators on X. If $\mathcal{B}$ is a Banach subalgebra of $B(X)$, a linear mapping $\delta : \mathcal{B} \to B(X)$ is called a bilocal derivation if for every $T \in \mathcal{B}$ and $u \in X$, there exists a derivation $\delta_{T,u} : \mathcal{B} \to B(X)$ (depending on T and u) such that $\delta(T)u = \delta_{T,u}(T)u$.

In [1], it was proved that every bilocal derivation of standard operator algebras is an inner derivation.

The main purpose of this paper is to show that if $\mathcal{A}$ is a reflexive algebra in reflexive Banach space X such that both $O_+ \neq O$ and $X_- \neq X$ in Lat$\mathcal{A}$, the invariant subspace lattice of $\mathcal{A}$, then every norm-continuous bilocal derivation of $\mathcal{A}$ into $B(X)$ is an inner derivation.

Motivated by the concept of topological reflexivity of linear space of operators, here we raise the concept of topologically algebraic bireflexivity as follows: If $\mathcal{B}$ is a subalgebra of $B(X)$ and $B(\mathcal{B}, B(X))$ denotes the algebra of all bounded linear operators from $\mathcal{B}$ to $B(X)$. For a subset $\mathcal{S}$ of $B(\mathcal{B}, B(X))$, we write

$$\mathrm{biref}_{at}(\mathcal{S}) = \{\varphi \in B(\mathcal{B}, B(X)) : \varphi(B)x \in \mathcal{S}(B)x, \forall B \in \mathcal{B} \text{ and } \forall x \in X\},$$

where $\mathcal{S}(B)x = \{\varphi(B)x : \varphi \in \mathcal{S}\}$. The set $\mathcal{S}$ is said to be topologically algebraically bireflexive if $\mathcal{S} = \mathrm{biref}_{at}(\mathcal{S})$. Therefore our result stated above can be restated that the

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set of all derivations of $\mathcal{A}$ into $B(X)$ is topologically algebraically bireflexive.

2. Preliminaries and notations

In what follows we denote by X a fixed complex Banach space. The usual notation $Lat\mathcal{B}$ will denote the lattice of invariant subspaces for a subset $\mathcal{B} \subseteq B(X)$, and $Alg\mathcal{L}$ will denote the algebra of bounded linear operators leaving invariant every member of a family $\mathcal{L}$ of subspaces. $\mathcal{B}$ is reflexive if $\mathcal{B} = \text{ref}\mathcal{B}$, where $\text{ref}\mathcal{B} = \{T \in B(X) : Tx \in [\mathcal{B}x], x \in X\}$, $[\cdot]$ denotes the norm closure.

For a lattice $\mathcal{L}$ of subspaces of X, if $N \in \mathcal{L}$, we denote $\wedge\{M \in \mathcal{L} : N \not\subseteq M\}$ by N_- and $\vee\{M \in \mathcal{L} : M \not\subseteq N\}$ by N_+.

For s subset $S \subseteq X$, $S^\perp = \{f \in X^* : f(S) = \{0\}\}$, where X^* is the dual space of X. If $x \in X$ and $f \in X^*$, the rank one operator $u \longmapsto f(u)x$ is denoted by $x \otimes f$.

Let $\mathcal{B}$ be a Banach subalgebra of $B(X)$, a linear mapping $\delta : \mathcal{B} \to B(X)$ has Property G if $\delta(T)\xi$ is in ran$T$ whenever ξ is in ker$T$, for all T in $\mathcal{B}$. Here, ran$T$ and ker$T$ stand for the range and kernal of T respectively.

Lemma 2.1[3]] *If $\mathcal{L}$ is a subspace lattice, then $x \otimes f \in \text{Alg}\mathcal{L}$ if and only if there exists an element $L \in \mathcal{L}$ such that $x \in L$ and $f \in (L_-)^\perp$.*

3. Bilocal derivations

In what follows $\mathcal{A}$ will be a reflexive algebra in Banach space X such that both $O_+ \neq O$ and $X_- \neq X$ in Lat$\mathcal{A}$ and $\delta : \mathcal{A} \to B(X)$ be a linear map.

Lemma 3.1 *If δ has Property G, given $N \in \text{Lat}\mathcal{A}$ with $N \neq O$, then for each nonzero $f \in (N_-)^\perp$ there exists a continuous linear functional λ_f^N on X such that*

$$\delta(x \otimes f)u = \lambda_f^N(u)x \text{ for any } u \in \ker f.$$

Proof Fix $f \in (N_-)^\perp$ with $f \neq 0$. As δ has Property G, $\delta(x \otimes f)$ maps ker$f$ into span$\{x\}$, so there is a continuous linear functional $\lambda_{f,x}^N$ on ker$f$ such that

$$\delta(x \otimes f)u = \lambda_{f,x}^N(u)x \text{ for any } u \text{ in } \ker f.$$

For any $y \in N$, we have

$$\delta((x + y) \otimes f)u = \lambda_{f,x+y}^N(u)(x + y).$$

On the other hand,

$$\delta((x + y) \otimes f)u = \delta(x \otimes f)u + \delta(y \otimes f)u = \lambda_{f,x}^N(u)x + \lambda_{f,y}^N(u)x.$$

Then we get that

$$(\lambda_{f,x+y}^N - \lambda_{f,x}^N)(u)x + (\lambda_{f,x+y}^N - \lambda_{f,y}^N)(u)y = 0.$$

Hence $\lambda_{f,x}^N = \lambda_{f,x+y}^N = \lambda_{f,y}^N$, so $\lambda_{f,x}^N$ is independent of x. We write $\lambda_f^N = \lambda_{f,x}^N$. Therefore λ_f^N can be uniquely extended to a continuous linear functional on X, still denoted by λ_f^N.

Lemma 3.2 *If δ has Property G, given $N \in \mathrm{Lat}\mathcal{A}$ with $N \neq O$, then for each $f \in (N_-)^\perp$, there exists $\lambda_f^N \in X^*$ and a linear mapping $B_f^N : N \to X$ such that*

$$\delta(x \otimes f) = x \otimes \lambda_f^N + B_f^N x \otimes f, \quad \forall x \in N.$$

Proof For any $u \in X \setminus \ker f$, define an operator $B_{u,f}^N : N \to X$ by

$$B_{u,f}^N x = (\delta(x \otimes f)u - \lambda_f^N(u)x)/f(u),$$

where λ_f^N is defined as in Lemma 3.1.

It is easy to see that $B_{u,f}^N$ is linear. Then we have that

$$\delta(x \otimes f)u = \lambda_f^N(u)x + f(u)B_{u,f}^N x, \quad \forall x \in N.$$

Take any $v \in X \setminus \ker f$ with $v \neq -u$ and $v \neq 0$, thus for any $x \in N$ we have that

$$\delta(x \otimes f)v = \lambda_f^N(v)x + f(v)B_{u,f}^N x,$$

and

$$\delta(x \otimes f)(u+v) = \lambda_f^N(u+v)x + f(u+v)B_{u+v,f}^N x.$$

This yields that $B_{u,f}^N = B_{u+v,f}^N - B_{v,f}^N$. Thus we may take $B_{u,f}^N = B_f^N$.

For any $u \in \ker f$, by Lemma 3.1, we have that $\delta(x \otimes f)u = \lambda_f^N(u)x$. Hence we get

$$\delta(x \otimes f)u = \lambda_f^N(u)x + f(u)B_f^N x, \quad \forall x \in N \quad \text{and} \quad u \in X,$$

i.e.,

$$\delta(x \otimes f)u = (x \otimes \lambda_f^N)u + (B_f^N \cdot x \otimes f)(u), \quad \forall x \in N \quad \text{and} \quad u \in X.$$

Therefore we obtain that

$$\delta(x \otimes f) = x \otimes \lambda_f^N + B_f^N x \otimes f, \quad \forall x \in N.$$

Lemma 3.3 *With δ as in Lemma 3.1, given $N \in \mathrm{Lat}\mathcal{A}$ with $N \neq O$, if f_1 and f_2 are linearly independent elements of $(N_-)^\perp$ and if $\lambda_{f_i}^N$, $B_{f_i}^N$, $i = 1,2$ are chosen to satisfy the conclusion of Lemma 3.2 for f_1 and f_2 respectively, then $B_{f_1}^N - B_{f_2}^N$ is a scalar multiple of the identity.*

Proof Clearly it suffices to show that for each $x \in N$, $(B_{f_1}^N - B_{f_2}^N)x$ is a scalar multiple of x. By Lemma 3.2, there exist $\lambda_{f_1+f_2}^N \in X^*$ and a linear mapping $B_{f_1+f_2}^N : N \to X$ such that

$$\delta(x \otimes (f_1 + f_2)) = x \otimes \lambda_{f_1+f_2}^N + B_{f_1+f_2}^N x \otimes (f_1 + f_2).$$

But we also have,

$$\begin{aligned}\delta(x \otimes (f_1 + f_2)) &= \delta(x \otimes f_1) + \delta(x \otimes f_2) \\ &= x \otimes \lambda_{f_1}^N + B_{f_1}^N x \otimes f_1 + x \otimes \lambda_{f_2}^N + B_{f_2}^N x \otimes f_2.\end{aligned}$$

Taking adjoints yields that

$$\lambda^N_{f_1+f_2} \otimes x + (f_1 + f_2) \otimes B^N_{f_1+f_2} x = (\lambda^N_{f_1} + \lambda^N_{f_2}) \otimes x + f_1 \otimes B^N_{f_1} + f_2 \otimes B^N_{f_2} x.$$

For every h in $\{x\}^\perp$, let the two sides of the above equation act on h, we obtain

$$h(B^N_{f_1+f_2} x)(f_1 + f_2) = h(B^N_{f_1} x) f_1 + h(B^N_{f_2} x) f_2,$$

and the linear independence of $\{f_1, f_2\}$ implies that $h(B^N_{f_1} x - B^N_{f_2} x) = 0$. But h is an arbitrary element of $\{x\}^\perp$. Thus $(B^N_{f_1} - B^N_{f_2})x$ is a scalar multiple of x, as desired. □

Lemma 3.4 *With δ as in Lemma 3.1. For each $N \in \operatorname{Lat}\mathcal{A}$ with $N \neq O$, there exists a linear mapping $B_N : N \to X$ with the property that for each functional $f \in (N_-)^\perp$ there is a functional $\lambda^N_f \in X^*$ such that*

$$\delta(x \otimes f) = x \otimes \lambda^N_f + B_N x \otimes f, \quad \text{for each} \quad x \in N. \tag{*}$$

Proof Fix a nonzero element $f_0 \in (N_-)^\perp$. By Lemma 3.3, there is a functional $\lambda^N_{f_0} \in X^*$ and a linear mapping $B^N_{f_0} : N \to X$ such that

$$\delta(x \otimes f_0) = x \otimes \lambda^N_{f_0} + B^N_{f_0} x \otimes f_0.$$

Let $B_N = B^N_{f_0}$. Therefore equation (*) holds for f_0 and also, holds for scalar multiples of f_0 by linearity.

On the other hand, if f and f_0 are linearly independent, let λ^N_f and B^N_f satisfy the conclusion of Lemma 3.2. Then by Lemma 3.3, $B^N_f - B = \mu I$ for some scalar μ. So for each $x \in N$, we have

$$\delta(x \otimes f) = x \otimes (\lambda^N_f + \mu f) + B_N x \otimes f$$

and equation (*) is satisfied for f with λ^N_f replaced by $\lambda^N_f + \mu f$.

Lemma 3.5 *With δ as in Lemma 3.1. Given $N \in \operatorname{Lat}\mathcal{A}$ with $N \neq O$, there exist linear mappings $B_N : N \to X$ and $C_N : (N_-)^\perp \to X^*$ such that*

$$\delta(x \otimes f) = x \otimes C_N f + B_N x \otimes f, \quad \forall x \in N \quad \text{and} \quad f \in (N_-)^\perp.$$

Proof Fix a nonzero vector $x \in N$ and fix $h \in X^*$ such that $h(x) = 1$. For each $f \in X^*$ the functional λ^N_f in (*) of Lemma 3.4 is given by

$$\lambda^N_f = (\delta(x \otimes f) - B_N x \otimes f)^*(h)$$

where B_N is as in (*). Therefore the map $f \to \lambda^N_f$ is linear. We call the map C_N, the proof is complete. □

Using the same proof in [4], we obtain

Lemma 3.6 *If X is reflexive and $\delta : \mathcal{A} \to B(X)$ is a norm-continuous linear mapping* having the Property G, then there exist operators $A, B \in B(X)$ such that $\delta(x \otimes f) = Ax \otimes f + x \otimes fB$ for any $x \in X$ and $f \in (X_-)^\perp$ or any $x \in O_+$ and $f \in X^*$.

Theorem 3.7 *If X is reflexive and $\delta : \mathcal{A} \to B(X)$ is a norm-continuous linear map having the Property G, then there exist $A, B \in B(X)$ such that $\delta(T) = AT + TB$ for any $T \in \mathcal{A}$.*

Proof By Lemma 3.6, we have that $\delta(R) = AR + RB$ where R is a finite linear combination of rank one operators of the form $x \otimes f$ with $x \in X$ and $f \in (X_-)^\perp$ or $x \in O_+$ and $f \in X^*$.

Let $\varphi : \mathcal{A} \to B(X)$ be defined by $\varphi(T) = AT + TB$ for every $T \in \mathcal{A}$, and let $\psi = \varphi - \delta$. Now we show that $\psi = 0$.

It is obvious that $\psi(R) = 0$ for every above-mentioned R.

Fix $T \in \mathcal{A}$ such that $\psi(T) \neq 0$, then there exists a nonzero vector $x \notin X_-$ with $\psi(T)x \neq 0$. And so there is a nonzero functional $f \in (X_-)^\perp$ such that $f(x) = 1$. And so $P = x \otimes f$ is a projection in $\mathcal{A}$.

(1) Let $g \notin (O_+)^\perp$. Then there is $y \in O_+$ such that $g(y) = 1$. we get that $Q = y \otimes g$ is also a projection in $\mathcal{A}$. For each $T \in \mathcal{A}$, by the hypothesis on δ we have that

$$Q\psi((I - Q)T(I - P))P = 0.$$

But $T - (I - Q)T(I - P)$ has the form of R, so

$$\psi(T - (I - Q)T(I - P)) = 0,$$

i.e.,

$$\psi(T) = \psi((I - Q)T(I - P)).$$

Hence $Q\psi(T)P = 0$ and so $g(\psi(T)x) = 0$ for each $g \notin (O_+)^\perp$.

(2) Let $g \in (O_+)^\perp$. If $\psi(T)x \in O_+$, then

$$g(\psi(T)x) = 0.$$

If $\psi(T)x \notin O_+$ and $g(\psi(T)x) \neq 0$, there is $h \notin (O_+)^\perp$ such that $h(\psi(T)x) = 0$. Then

$$g + h \notin (O_+)^\perp \text{ and } (g + h)(\psi(T)x) \neq 0.$$

This is in contradiction with (1).

It follows that $g(\psi(T)x) = 0$ for every $g \in X^*$. And so $\psi(T)x = 0$, a contradiction. □

Theorem 3.8 *If X is reflexive and $\delta : \mathcal{A} \to B(X)$ is a norm-continuous bilocal derivation, then δ is an inner derivation.*

Proof Since δ is a bilocal derivation of $\mathcal{A}$ into $B(X)$, and by Theorem 3.1 of [4], every derivation of $\mathcal{A}$ into $B(X)$ is inner, then δ satisfies the condition of Theorem 3.7. Hence there exist operators $A, B \in B(X)$ such that $\delta(T) = AT + TB$ for any $T \in \mathcal{A}$. Then $A + B = \delta(I) = 0$, thus $B = -A$, so δ is an inner derivation. □

Corollary 3.9 *If X is a reflexive Banach space, then the set of all derivations of $\mathcal{A}$ into $B(X)$ is topologically algebraically bireflexive.*

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References:

[1] ZHU Jun, XIONG Chang-ping. *Bilocal derivations of standard operator algebras* [J]. Proc. Amer. Math. Soc., 1997, **125**: 1367–1370.

[2] LARSON D R. *Reflexivity, algebraic reflexivity and linear interpolation* [J]. Amer. J. Math., 1988, **110**: 283–299.

[3] LONGSTAFF W E. *Strongly reflexive subspace lattices* [J]. J. London. Math. Soc., 1975, **11**: 491–498.

[4] JING Wu. *Local derivations of reflexive algebras* [J]. Proc. Amer. Math. Soc., 1997, **125**: 869–873.

[5] SPIVACK M. *Certain properties of derivations* [J]. Proc. Amer. Math. Soc., 1987, **99**: 712–718.

[6] LARSON D R, SOUROUR A R. *Local derivations and local automorphisms of B(X)* [J]. Proc. Sym. Pure Math., 1990, **51**: 187–194.

自反代数的双局部导子

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摘 要: 证明了自反 Banach 空间 X 中自反代数 $\mathcal{A}$ 到 $B(X)$ 的导子集合是拓扑代数双自反的.