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Oscillation Theorems of Second Order Nonlinear Neutral Functional Differential Equations *

WANG Qi-ru[1,2], *CHENG Shi-hui*[3]
(1. Dept. of Math., Zhongshan University, Guangzhou 510275, China;
2. Inst. of Math., Acad. of Math. & Sys. Sci., Chinese Acad. of Sci., Beijing 100080, China;
3. Dept. of Math., Henan College of Education, Zhengzhou 450000, China)

Abstract: This paper, we discuss a class of second order nonlinear neutral differential equations with variable coefficients and variable deviations. Sharp conditions are established for all bounded solutions of the equations to be oscillatory. Linearized oscillation criteria of the equations are also given.

Key words: neutral equation; "limit" equation; oscillation.

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1. Introduction

Consider the second order nonlinear neutral differential equation

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}[y(t)-\sum_{i=1}^{m}P_i(t)y(t-\tau_i)]=Q(t)\prod_{j=1}^{n}[f_j(y(t-\sigma_j(t)))]^{\alpha_j},t\geq t_0, \tag{1}$$

where $\tau_i>0, P_i(t), Q(t), \sigma_j(t)\in C([t_0,\infty),R^+), f_j\in C(R,R), yf_j(y)>0 (y\neq 0), \alpha_j\geq 0$ is a rational number whose denominator is odd, $\lim_{t\to\infty}(t-\sigma_j(t))=\infty, i=1,2,\ldots,m, j=1,2,\ldots,n$, and $\sum_{j=1}^{n}\alpha_j=1$.

Definition 1 *If*

$$\lim_{t\to\infty}P_i(t)=p_i\,(i=1,2,\ldots,m),\ \lim_{t\to\infty}Q(t)=q, \tag{2}$$

$$\lim_{y\to 0}f_j(y)/y=1\ (j=1,2,\ldots,n), \tag{3}$$

then we call the equation

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}[y(t)-\sum_{i=1}^{m}p_iy(t-\tau_i)]=q\prod_{j=1}^{n}[y(t-\sigma_j)]^{\alpha_j} \tag{4}$$

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Biography: WANG Qi-ru (1965-), male, born in Zhenping county, Henan province. Ph.D., associate professor.

is the "limit" equation of (1) when $\sigma_j(t) = \sigma_j$ is a constant ($j = 1, 2, \ldots, n$).

Recently a linearized oscillation theory has been developed in [1-5, etc.] for nonlinear differential equations. Roughly speaking, it has been proved that, under appropriate hypotheses, certain nonlinear differential equations have the same oscillatory character as the associated linear equations. Qian and Yu[5] studied the oscillation of all bounded solutions of the special cases of (1) where $m = n = 1$, where their argument depends on the identical transformations of the equations. However, there is no similar transformation of (1). So in this paper, by using a method which is different from [5, 6], we establish sufficient conditions for all bounded solutions of (1) to be oscillatory, and the conditions are sharp in the sense that when the coefficients $P_i(t)$ and $Q(t)$ are constants ($i = 1, 2, \ldots, m$), $f_j(y) = y$ and each delay $\sigma_j(t)$ is a constant ($j = 1, 2, \ldots, n$) they are also necessary. Then, we give the linearized oscillation criteria of (1), that is, we establish the oscillation criteria for (1) based on the oscillation of "limit" equation (4).

As usual, a solution of (1) is called oscillatory if it has arbitrarily large zeros and nonoscillatory if it is eventually positive or eventually negative.

In what follows, we shall always assume that $f_j (j = 1, \ldots, n)$ in (1) guarantee the existence of solutions of (1) on $[t_0, \infty)$. For convenience, we use the convention that all the inequalities involving t hold eventually.

2. Basic Lemmas

Lemma 1 *In (1), assume that*

(H_1) Either $\sum_{i=1}^{m} P_i(t) \leq 1$ holds eventually, or each of $P_i(t) (i = 1, 2, \ldots, m)$ is bounded and there exists a $\tau > 0$, natural numbers $k_i (i = 1, 2, \ldots, m)$ and a $t^* \geq t_0$ such that

$$\tau_i = k_i \tau (i = 1, 2, \ldots, m), \quad \sum_{i=1}^{m} P_i(t^* + k\tau) \leq 1, k = 0, 1, 2, \ldots;$$

(H_2) there is a $q > 0$ such that $Q(t) \geq q$.
If $y(t)$ is an eventually bounded positive solution of (1) and

$$z(t) = y(t) - \sum_{i=1}^{m} P_i(t) y(t - \tau_i), \tag{5}$$

then we have

$$z''(t) > 0, z'(t) < 0, z(t) > 0; \tag{6}$$

$$\lim_{t \to \infty} z'(t) = \lim_{t \to \infty} z(t) = 0. \tag{7}$$

Proof Without loss of generality, suppose that $\tau_1 < \tau_2 < \ldots < \tau_m$. It follows from (1) that $z''(t) > 0$ on $[t_1, \infty)$ for some sufficiently large $t_1 \geq t_0$. From (H_1) and (5) we see that $|z(t)|$ is bounded. Hence

$$\lim_{t \to \infty} z'(t) = \lim_{t \to \infty} z(t)/t = 0 \quad \text{and} \quad z'(t) < 0,$$

which implies that $z(t)$ is eventually decreasing. Therefore $\lim_{t\to\infty} z(t) = \beta$ exists and is finite. Now we claim that $\beta = 0$.

Following the procedure of the proof of [7, Lemma 1], we can prove that $\beta \geq 0$. Now we claim that $\liminf_{t\to\infty} y(t) = 0$. Otherwise, if $\liminf_{t\to\infty} y(t) = b_1 > 0$, then there exist a $t_2 \geq t_1$ and a $b_2 > 0$ such that $y(t - \sigma_j(t)) \in [b_1/2, b_2]$ for $t \geq t_2, j = 1, 2, \ldots, n$. As $f_j \in C(R, R)$ and $f_j(y) > 0 (y > 0), f_j(y)$ attains a minimum $h_j > 0$ on $[b_1/2, b_2]$, that is,

$$f_j(y(t - \sigma_j(t))) \geq h \triangleq \min_{1\leq j\leq n} h_j, t \geq t_2, j = 1, 2, \ldots, n.$$

From (1) it follows that $z''(t) \geq qh > 0, t \geq t_2$, which implies that $z'(t) \to \infty$ and $z(t) \to \infty$ as $t \to \infty$. This contradicts the boundedness of $z(t)$ and so the claim holds.

Also from (5) it follows that $z(t) \leq y(t)$, and $\liminf_{t\to\infty} z(t) \leq \liminf_{t\to\infty} y(t) = 0$.

Thus we have $\lim_{t\to\infty} z(t) = 0$, and so $z(t) > 0$.

The proof of Lemma 1 is completed. □

From Lemma 1 and Lemma 1 of [8], we have the following lemma.

Lemma 2 *In (1), suppose that* (H_2) *holds and*

$$\lim_{t\to\infty} P_i(t) = p_i \, (i = 1, 2, \ldots, m), \sum_{i=1}^{m} p_i < 1. \tag{8}$$

If $y(t)$ *is an eventually bounded positive solution of (1), then the results of Lemmal 1 hold and* $\lim_{t\to\infty} y(t) = 0$.

Remark 1 The condition (H_1) in Lemma 1 allows $\sum_{i=1}^{m} P_i(t) - 1$ to be oscillatory. When $m = 1$, it becomes the condition (i) of Theorem 3.1 in [6].

Remark 2 In Lemmas 1 and 2, if $y(t)$ is an eventually bounded negative solution of (1), then the relevant results hold.

3. Main results

Theorem 1 *In (1), assume that* (H_1) *holds and*

(H_3) *there exist positive constants* K_1 *and* K_2 *such that* $K_1 \leq Q(t) \leq K_2$;

(H_4) *there exist positive constants* M_j *and* N_j *such that* $N_j y^2 \leq y f_j(y) \leq M_j y^2, y \in R, j = 1, 2, \ldots, n$;

(H_5) $\sigma_j(t - \tau_i) = \sigma_j(t), i = 1, 2, \ldots, m, j = 1, 2, \ldots, n$, *and there exists a* $\sigma > 0$ *such that* $\min_{1\leq j\leq n, t\geq t_0}\{\sigma_j(t)\} \geq \sigma$;

(H_6) *there exists a* $T \geq t_0$ *such that*

$$\inf_{t\geq T, \lambda>0} \frac{1}{\lambda^2} Q(t)(\prod_{j=1}^{n} N_j^{\alpha_j})\{\exp[\lambda \sum_{j=1}^{n} \alpha_j \sigma_j(t)] +$$
$$\lambda^2 (\prod_{j=1}^{n} M_j^{-\alpha_j}) \sum_{i=1}^{m} [Q(t - \tau_i)]^{-1} \exp(\lambda \tau_i) \prod_{j=1}^{n} [P_i(t - \sigma_j(t))]^{\alpha_j}\} > 1,$$

then all bounded solutions of (1) are oscillatory.

Proof If there exists a bounded nonoscillatory solution $y(t)$, we suppose that $y(t)$ is eventually positive (if $y(t)$ is eventually negative, it can be treated in a similar way). From (5) and Lemma 1, we see that (6) and (7) hold. From (5) it follows that $z(t) \le y(t)$. Also from (1), (6) and (H_3)–(H_5), we have

$$\begin{aligned} z''(t) &\geq K_1(\prod_{j=1}^{n} N_j^{\alpha_j}) \prod_{j=1}^{n}[y(t-\sigma_j(t))]^{\alpha_j} && (9) \\ &\geq K_1(\prod_{j=1}^{n} N_j^{\alpha_j}) \prod_{j=1}^{n}[z(t-\sigma_j(t))]^{\alpha_j} \\ &\geq K_1(\prod_{j=1}^{n} N_j^{\alpha_j}) z(t-\sigma) && (10) \\ &> K_1(\prod_{j=1}^{n} N_j^{\alpha_j}) z(t) \end{aligned}$$

for $t \ge t_1$ ($\ge T$, sufficiently large).

Define $\Lambda = \{\lambda > 0 : z''(t) > \lambda^2 z(t) \quad \text{eventually}\}$, then $\sqrt{K_1 \prod_{j=1}^{n} N_j^{\alpha_j}} \in \Lambda$, i.e., Λ is nonempty. Next we show that Λ is bounded above. In fact, from (10) and in a similar way to [5] we can prove that

$$z(t) \geq \delta^4 z(t-\sigma), \quad t \geq t_1 + 2\sigma, \tag{11}$$

where $\delta = \sigma^2 K_1 \left(\prod_{j=1}^{n} N_j^{\alpha_j}\right)/8$. On the other hand, let

$$\Phi(t) = \prod_{j=1}^{n}[y(t-\sigma_j(t))]^{\alpha_j}.$$

From (6), (7) and (9) it is not difficult to see that $\liminf_{t\to\infty} \Phi(t) = 0$. Then there exists a sequence $\{ s_k\}_{k=1}^{\infty}$ such that

(i) $s_k \ge t_1 + 2\sigma, k = 1, 2, \ldots,$ and $s_k \to \infty (k \to \infty)$;

(ii) $\Phi(s_k) = \min\{\Phi(s) : t_1 \le s \le s_k\}, k = 1, 2, \ldots$.

Integrating (9) from $u (\le s_k)$ to s_k, and then integrating on u from $s_k - \sigma$ to s_k we have

$$\begin{aligned} z(s_k - \sigma) &> K_1(\prod_{j=1}^{n} N_j^{\alpha_j}) \int_{s_k-\sigma}^{s_k} \int_{u}^{s_k} \Phi(v) dv du \\ &\geq \sigma^2 K_1(\prod_{j=1}^{n} N_j^{\alpha_j}) \Phi(s_k)/2, \end{aligned}$$

that is,

$$\Phi(s_k) < \beta z(s_k - \sigma), k = 1, 2, \ldots, \tag{12}$$

where $\beta = 2\left(\prod_{j=1}^{n} N_j^{-\alpha_j}\right)/\sigma^2 K_1$. From (1), (11), (12) and (H_3), it follows that

$$\begin{aligned} z''(s_k) &\leq K_2(\prod_{j=1}^{n} M_j^{\alpha_j}) \prod_{j=1}^{n}[y(s_k - \sigma_j(s_k))]^{\alpha_j} \\ &< K_2(\prod_{j=1}^{n} M_j^{\alpha_j}) \beta z(s_k - \sigma) < K_2 \beta (\prod_{j=1}^{n} M_j^{\alpha_j}) \delta^{-4} z(s_k), k = 1, 2, \ldots, \end{aligned}$$

which implies that Λ is bounded above. Set $\lambda_0 = \sup\Lambda$, clearly $\lambda_0 > 0$ and

$$z''(t) \geq \lambda_0^2 z(t). \tag{13}$$

This implies that $[z'(t)+\lambda_0 z(t)]e^{-\lambda_0 t}$ is eventually nondecreasing, and then $z'(t)+\lambda_0 z(t) \leq 0$ from (7). Let $w(t) = z(t)e^{\lambda_0 t}$, then $w'(t) \leq 0$. From (1) and (13) it follows that

$$\Phi(t) \geq \lambda_0^2 (Q(t)\prod_{j=1}^{n} M_j^{\alpha_j})^{-1} z(t). \tag{14}$$

From (1), (5), (14), (H_5) and the Hölder inequality we have

$$\begin{aligned}
z''(t) &\geq Q(t)(\prod_{j=1}^{n} N_j^{\alpha_j})\prod_{j=1}^{n}[y(t-\sigma_j(t))]^{\alpha_j} \\
&= Q(t)(\prod_{j=1}^{n} N_j^{\alpha_j})\prod_{j=1}^{n}[z(t-\sigma_j(t)) + \sum_{i=1}^{m} P_i(t-\sigma_j(t))y(t-\sigma_j(t)-\tau_i)]^{\alpha_j} \\
&\geq Q(t)(\prod_{j=1}^{n} N_j^{\alpha_j})\{\prod_{j=1}^{n}[z(t-\sigma_j(t))]^{\alpha_j} + \sum_{i=1}^{m}\Phi(t-\tau_i)\prod_{j=1}^{n}[P_i(t-\sigma_j(t))]^{\alpha_j}\} \\
&\geq Q(t)(\prod_{j=1}^{n} N_j^{\alpha_j})\{\prod_{j=1}^{n}[z(t-\sigma_j(t))]^{\alpha_j} + \\
&\quad \sum_{i=1}^{m}\lambda_0^2(Q(t-\tau_i)\prod_{j=1}^{n} M_j^{\alpha_j})^{-1} z(t-\tau_i)\prod_{j=1}^{n}[P_i(t-\sigma_j(t))]^{\alpha_j}\} \\
&= Q(t)(\prod_{j=1}^{n} N_j^{\alpha_j})\{\prod_{j=1}^{n}[w(t-\sigma_j(t))\exp(-\lambda_0(t-\sigma_j(t)))]^{\alpha_j} + \\
&\quad \lambda_0^2(\prod_{j=1}^{n} M_j^{-\alpha_j})\sum_{i=1}^{m}[Q(t-\tau_i)]^{-1} w(t-\tau_i)\exp(-\lambda_0(t-\tau_i))\prod_{j=1}^{n}[P_i(t-\sigma_j(t))]^{\alpha_j}\} \\
&\geq Q(t)(\prod_{j=1}^{n} N_j^{\alpha_j})\{\exp[\lambda_0\sum_{j=1}^{n}\alpha_j\sigma_j(t)] + \\
&\quad \lambda_0^2(\prod_{j=1}^{n} M_j^{-\alpha_j})\sum_{i=1}^{m}[Q(t-\tau_i)]^{-1}\exp(\lambda_0\tau_i)\prod_{j=1}^{n}[P_i(t-\sigma_j(t))]^{\alpha_j}\} z(t),
\end{aligned}$$

which implies that

$$\begin{aligned}
&\inf_{t\geq t_1} Q(t)(\prod_{j=1}^{n} N_j^{\alpha_j})\{\exp[\lambda_0\sum_{j=1}^{n}\alpha_j\sigma_j(t)] + \\
&\quad \lambda_0^2(\prod_{j=1}^{n} M_j^{-\alpha_j})\sum_{i=1}^{m}[Q(t-\tau_i)]^{-1}\exp(\lambda_0\tau_i)\prod_{j=1}^{n}[P_i(t-\sigma_j(t))]^{\alpha_j}\} \leq \lambda_0^2.
\end{aligned}$$

Since $t_1 \geq T$ and $\lambda_0 > 0$, the above inequality contradicts (H_6).

The proof of Theorem 1 is completed. □

Theorem 2 *In (1), assume that (8), (H_3), (H_5) and (H_6) hold, and*

$$\liminf_{y\to 0} f_j(y)/y \geq N_j > 0, \quad \limsup_{y\to 0} f_j(y)/y \leq M_j < \infty, j = 1,2,\ldots,n.$$

Then all bounded solutions of (1) are oscillatory.

Note that if there exists a bounded nonoscillatory solution $y(t)$ of (1), then by Lemma 2 we know that $\lim_{t\to\infty} y(t) = 0$. Hence the essential parts of (H_4) hold and the proof of Theorem 2 is similar to that of Theorem 1.

Corollary 1 *In (4), suppose that $p_i \geq 0, q > 0, \sigma_j > 0$, τ_i and α_j are assumed as in (1), $i = 1,2,\ldots,m, j = 1,2,\ldots,n$, and $\sum_{i=1}^{m} p_i \leq 1$. Then all bounded solutions of (4) are oscillatory if and only if*

$$F(\lambda) = -\lambda^2 + \lambda^2 \sum_{i=1}^{m} p_i \exp(\lambda\tau_i) + q\exp(\lambda\sum_{j=1}^{n}\alpha_j\sigma_j) > 0, \lambda > 0.$$

Proof The sufficiency part can be proved from Theorem 1 with $M_j = N_j = 1 (j = 1,2,\ldots,n)$ immediately. For the necessity part, if there is a $\lambda_1 > 0$ such that $F(\lambda_1) \leq 0$. Since $F(0) = q > 0$, there exists a $\lambda_0 \in (0,\lambda_1]$ such that $F(\lambda_0) = 0$. It is easy to check that $y(t) = \exp(-\lambda_0 t)$ is a bounded nonoscillatory solution of (4).

The proof is completed. □

Remark 3 Corollary 1 implies that the conditions of Theorems 1 and 2 are sharp.

In the following, we establish the linearized oscillation criteria for (1). First we give a lemma which will be needed in the proof of Theorem 3. Its proof is similar to that of [5, Lemma 4] and is omitted here.

Lemma 3 *In (4), assume that $p_i > 0, q > 0, \sigma_j > 0$, τ_i and α_j are assumed as in (1), $i = 1,2,\ldots,m, j = 1,2,\ldots,n$, and $\sum_{i=1}^{m} p_i \leq 1$. If all bounded solutions of (4) are oscillatory, then there is a $\varepsilon_0 > 0$ such that all bounded solutions of the equation*

$$\frac{d^2}{dt^2}[y(t) - \sum_{i=1}^{m}(p_i - \varepsilon)y(t-\tau_i)] = (q-\varepsilon)\prod_{j=1}^{n}[y(t-\sigma_j)]^{\alpha_j}$$

are oscillatory for every $\varepsilon \in [0,\varepsilon_0]$.

Theorem 3 *In (1), suppose that (2), (3) and (H_4) hold. Also suppose that $\sum_{i=1}^{m} p_i < 1, q > 0, \sigma_j(t) \geq \sigma_j > 0 (j = 1,2,\ldots,n)$. If all bounded solutions of (4) are oscillatory, then all bounded solutions of (1) are also oscillatory.*

Proof We consider two cases.

(i) $p_i = 0, i = 1,2,\ldots,m$. It follows from Corollary 1 that all bounded solutions of (4) are oscillatory if and only if

$$-\lambda^2 + q\exp(\lambda\sum_{j=1}^{n}\alpha_j\sigma_j) > 0, \lambda > 0.$$

From (3) we see that $M_j = N_j = 1 (j = 1, 2, \ldots, n)$. Then for every $\lambda > 0$, we have

$$\begin{aligned}
&\liminf_{t\to\infty} \frac{1}{\lambda^2} Q(t)\left(\prod_{j=1}^{n} N_j^{\alpha_j}\right)\{\exp[\lambda \sum_{j=1}^{n} \alpha_j \sigma_j(t)] + \\
&\quad \lambda^2 \left(\prod_{j=1}^{n} M_j^{-\alpha_j}\right) \sum_{i=1}^{m} [Q(t-\tau_i)]^{-1} \exp(\lambda\tau_i) \prod_{j=1}^{n} [P_i(t-\sigma_j(t))]^{\alpha_j}\} \\
&\geq \frac{1}{\lambda^2} q \exp(\lambda \sum_{j=1}^{n} \alpha_j \sigma_j) > 1,
\end{aligned}$$

which implies that the conditions of Theorem 2 are satisfied. Hence, all bounded solutions of (1) are oscillatory.

(ii) $0 < \sum_{i=1}^{m} p_i < 1$. Set $I = \{i : p_i > 0, i = 1, 2, \ldots, m\}$, then I is nonempty. From Lemma 3, if all bounded solutions of (4) are oscillatory, then there is an $\varepsilon_0 > 0$ such that all bounded solutions of the equation

$$\frac{d^2}{dt^2}[y(t) - \sum_{i\in I}(p_i - \varepsilon) y(t-\tau_i)] = (q-\varepsilon) \prod_{j=1}^{n} [y(t-\sigma_j)]^{\alpha_j}$$

are oscillatory for every $\varepsilon \in [0, \varepsilon_0]$. It follows from Corollary 1 that

$$\sum_{i\in I}(p_i - \varepsilon)\exp(\lambda\tau_i) + \frac{1}{\lambda^2}(q-\varepsilon)\exp(\lambda \sum_{j=1}^{n} \alpha_j\sigma_j) > 1, \lambda > 0$$

for every $\varepsilon \in [0, \varepsilon_0]$. From (2) it follows that

$$\lim_{t\to\infty} \frac{Q(t)}{Q(t-\tau_i)} \prod_{j=1}^{n} [P_i(t-\sigma_j(t))]^{\alpha_j} = p_i > 0,\ i \in I, \quad \lim_{t\to\infty} Q(t) = q > 0.$$

So for every $\varepsilon \in (0, \varepsilon_0]$, there is a sufficient large $T \geq t_0$ such that

$$\frac{Q(t)}{Q(t-\tau_i)} \prod_{j=1}^{n} [P_i(t-\sigma_j(t))]^{\alpha_j} \geq p_i - \varepsilon, i \in I, \quad Q(t) \geq q - \varepsilon, \quad t \geq T.$$

From (3) we have $M_j = N_j = 1 (j = 1, 2, \ldots, n)$. Then

$$\begin{aligned}
&\inf_{t\geq T, \lambda>0} \left(\prod_{j=1}^{n} N_j^{\alpha_j}\right)\{\frac{1}{\lambda^2} Q(t) \exp[\lambda \sum_{j=1}^{n} \alpha_j\sigma_j(t)] + \\
&\quad \left(\prod_{j=1}^{n} M_j^{-\alpha_j}\right) \sum_{i=1}^{m} \frac{Q(t)}{Q(t-\tau_i)} \exp(\lambda\tau_i) \prod_{j=1}^{n} [P_i(t-\sigma_j(t))]^{\alpha_j}\} \\
&\geq \inf_{t\geq T, \lambda>0} \{\sum_{i\in I}(p_i-\varepsilon)\exp(\lambda\tau_i) + \frac{1}{\lambda^2}(q-\varepsilon)\exp(\lambda\sum_{j=1}^{n}\alpha_j\sigma_j)\} > 1
\end{aligned}$$

which implies that the conditions of Theorem 2 are satisfied. Hence, all bounded solutions of (1) are oscillatory.

The proof of Theorem 3 is completed. □

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二阶非线性中立型泛函微分方程的振动定理

王 其 如 [1,2], 程 世 辉 [3]

(1. 中山大学数学系, 广东 广州 510275; 2. 中科院数学与系统科学研究院数学研究所, 北京 100080;
3. 河南教育学院数学系, 河南 郑州 450000)

摘 要: 本文研究一类具有变系数和变偏差的二阶非线性中立型微分方程, 得到了这类方程的所有有界解都振动的 Sharp 条件, 并给出了这类方程的线性化振动准则.