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Weakly Compact Sets and Orlicz Spaces *

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Abstract: In this paper, We give the simple criteria of weakly compact sets in L_1 and l_1, which perfects Anto's result [1]. Also as a corollary, we get Shur's theorem. In view of weak compactness, we give another proof of the reflexivity of Orlicz spaces.

Key words: Compact sets; weak topology; Orlicz space.

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Let X be a Banach space and X^* its dual space, $B(X)$ be the unit ball of a Banach space X. A set D in Banach space X is said to be (weakly) compact provided that every sequence in D has a (weakly) convergent subsequence [7]. A set D in Banach space X is said to be relatively (weakly) compact provided that the (weak)closure of D is (weakly)compact. In 1960 [1], Anto gave the powerful criterion of weakly compact sets for Orlicz spaces, but it fails for L_1 and l_1. In this paper we give them for L_1 and l_1, so that Shur's theorem follows directly as a corollary.

Let us recall that $L_1(G,\sum,\mu)=\{x(t):\int_G|x(t)|\mathrm{d}\mu<\infty\}$ where $(G,\sum,\mu)$ is a measurable space with $\mu G<\infty$ and $l_1=\{x=x(i):\sum_{i=1}^{\infty}|x(i)|<\infty\}$.

Theorem 1 *For a set A in $L_1(G,\sum,\mu)$ with $\mu G<\infty$, A is relatively weakly compact if and only if A is absolutely equi-integrable, i.e., for all $\varepsilon>0$, there exists $\delta>0$ such that for all $E\subseteq G$ with $\mu E<\delta$, we have that for all $x\in A$, $\int_E|x(t)|\mathrm{d}\mu<\varepsilon$.*

Proof Without loss of generality, we assume that $G=[0,1]$ and μ is a Lebesgue measure.

Suppose that A is not absolutely equi-integrable,i.e., there exists $\varepsilon_0>0$ such that for each integer n we have $x_n\in A$ and $E_n\subset G$ with $\mu E_n\leq\frac{1}{n}$ and $\int_{E_n}|x_n(t)|dt\geq\varepsilon_0$, Obviously, all subsequence of $\{x_n\}$ are not absolutely equi-integrable. Since A is relatively weak compact,by Eberlein-Smulian Theorem, there exists a weakly convergent subsequence of $\{x_n\}$ that converges weakly to some $x\in L_1$. Without loss of generality, we assume that $\{x_n\}$ converges weakly to x. By Legesgue's Theorem (P172 of [5]), it

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follows that $\{x_n - x\}$ is absolutely equi-integrable, so does $\{x_n\}$(since x is absolutely equi-integrable), a contradiction.

Conversely, for any $A \subset L_1[0,1]$ for which A is absolutely equi-integrable. Since $L_1[0,1] \subset (C[0,1])^*$ and $C[0,1]$ is separable, by Theorem 1.1.4 of [11] and Banach-Alaoglu-Bourbaki Theorem, $(U(X^*), \sigma(X^*, X))$ is metrizated and compact. At first, we see that A is bounded. In fact, take $\delta > 0$, and split $[0,1]$ into parts $E_1, ..., E_n$ such that $\mu(E_i) \le \delta$ then for all $x \in A$

$$\|x\|_1 = \int_0^1 |x(t)|\mathrm{d}t = \sum_{i=1}^n \int_{E_i} |x(t)|\mathrm{d}t \le n.$$

For any $\{x_n\} \subset A$. Since A is sequential $\sigma(X^*, X)$ compact, there exists a convergent subsequence, still written as $\{x_n\}$, such that $w - \lim_{n\to\infty} x_n = x$ for some $x \in L_1([0,1])$ in the topology $\sigma(X^*, X)$ where $X^* = L_1([0,1]), X = C[0,1]$. In the following, we shall show that $\{x_n\}$ converges weakly in the topology $\sigma(L_1, L_\infty)$. In fact for any $f \in L_1^* = L_\infty$, for any $\varepsilon > 0$, and $\delta > 0$, by Lusin's theorem, there exists $g \in C[0,1]$ such that $\|g\|_\infty = \|f\|_\infty$, $\mu\{f \ne g\} \le \delta$ and $\int_{f\ne g} |f - g|\mathrm{d}t < \varepsilon$. Since $\lim_{n\to\infty}\langle g, x_n - x\rangle = 0$, there exists n_0, for $m, n \ge n_0$, $\langle g, x_n - x\rangle \le \varepsilon$. Since A is absolutely equi-integrable, there exists $\delta > 0$ such that for all subsets E with $\mu E < \delta$ and all $z \in A$, $\int_E |z(t)|\mathrm{d}t < \varepsilon$. So

$$\begin{aligned}|\langle f, x_n - x\rangle| &\le |\langle g, x_n - x\rangle| + |\langle f - g, x_n - x\rangle| \\ &< \varepsilon + 2\|f\|_\infty\Big(\int_{f\ne g} |x_n(t)|\mathrm{d}t + \int_{f\ne g} |x(t)|\mathrm{d}t\Big) \le \varepsilon + 2\|f\|_\infty 2\varepsilon = (1 + 4\|f\|_\infty)\varepsilon.\end{aligned}$$

By the arbitrariness of f, we see that $w - \lim x_n = x$. □

Next, we discuss the relatively weakly compact sets in Orlicz spaces. Let $\Re$ be the set of all real numbers. A function $M : \Re \to \Re_+$ is called an Orlicz function if M is convex, and $\lim_{u\to 0} \frac{M(u)}{u} = 0$ $\lim_{u\to\infty} \frac{M(u)}{u} = \infty$. A complemented function N of M is defined in the sense of Young's by $N(v) = \sup_{u\in\Re}\{uv - M(u)\}$. It is known that if M is an Orlicz function and its complemented function N is also an Orlicz function. M is said to satisfy $\triangle_2(\delta_2)$-condition for large (small) u (simply write $M \in \triangle_2(\delta_2)$) if for some K and $u_0 > 0$, $M(2u) \le KM(u)$ as $|u| \ge u_0(|u| \le u_0)$. Let $(G, \sum, \mu)$ be a measurable space with $\mu G < \infty$. For a measurable function $x(t)$ we call $\rho_M(x) = \int_G M(x(t))\mathrm{d}\mu$ the modular of x. For a scalar sequence $x = \{x(i)\}$ we call $\rho_M(x) = \sum_{i=1}^\infty M(x_i)$ the modular of x. The Orlicz space L_M generated by M is a Banach space

$$L_M = \{x : \exists\lambda > 0, \rho_M(\lambda x) < \infty\}$$

equipped with Orlicz norm

$$\|x\|^0 = \inf_{k>0} \frac{1}{k}\{1 + \rho_M(kx)\} = \frac{1}{k}\{1 + \rho_M(kx)\}, \forall k \in [k^*, K^{**}].$$

where $k^* = \inf\{k : \rho_N(P(kx)) \ge 1\}$,$k^{**} = \sup\{k : \rho_N(P(kx)) \le 1\}$, or equipped with Luxemburg norm

$$\|x\| = \inf\{\lambda > 0 : \rho_M(\frac{x}{\lambda}) \le 1\}.$$

(more detail can be seen in [KR,WC]).

Since the compactness is invariable topologically, we only discuss it for one norm.

Theorem 2 *In Orlicz function space L_M, all bounded sets are relatively L_N-weakly sequential compact if and only if $N \in \triangle_2$.*

Proof Sufficiency. For a bounded set A, without loss of generality, we can assume $A \subset B(L_M)$. For all $x \in A, \rho_M(x) \leq 1$. Since $N \in \triangle_2$, i.e., there exist $u_0 > 0, 1 > \beta > 0$ and $a < 1$ such that for all $|u| \geq u_0, M(\beta u) \leq a\beta M(u)$. Thus

$$\lim_{n\to\infty} \sup_{x\in A} \frac{\rho_M(\beta^n x)}{\beta^n} \leq \lim_{n\to\infty} \sup_{x\in A}\Big[\frac{\rho_M(\beta^n u_0\chi_x)}{\beta^n} + a^n\rho_M(x)\Big]$$
$$\leq \lim_{n\to\infty} \Big[\frac{M(\beta^n u_0)\mu G}{\beta^n} + a^n\Big] \to 0.$$

By [1], we get that A is relatively L_N-weakly compact.

Necessity. Now we assume that the Orlicz space L_M is equipped with Orlicz norm. Suppose that $N \notin \triangle_2$, then there exist $v_n \nearrow \infty$ and disjoint subsets $G_n \subset G$ such that

$$N((1+\frac{1}{n})v_n) > 2^n N(v_n), \quad N(v_n)\mu G_n = \frac{1}{2^n}.$$

Let

$$v(t) = \begin{cases} v_n & \text{if } t \in G_n, n = 1, 2, \cdots \\ 0 & \text{otherwise,} \end{cases}$$

Then $\rho_N(v) = \sum_{n=1}^{\infty} N(v_n)\mu G_n = 1 < \infty$. Let $v_n = v_n\chi|_{G_n}$. We see that $1 \geq \|v\|_{(N)} \geq \frac{1}{1+\frac{1}{n}}$. By [9], there exists $x_n \in L_M$ with $\|x_n\|^0 = 1$, $u_n = u_n\chi|_{G_n}$ and $\langle v_n, x_n\rangle = \|v_n\|_{(N)}$. Let $A = \{x_n\}$, then $A \subset B(L_M)$, but A is not L_N-weakly compact. In fact, take $\sigma_n = \mu G_n$ then $\sigma_n \to 0$. From

$$\lim_{n\to\infty} \sup_m \langle v|_{G_n}, x_m\rangle \geq \lim_{n\to\infty} \langle v_n, x_n\rangle = \lim_{n\to\infty} \|v_n\|_{(N)} = \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}} = 1,$$

by [1], A is not relatively L_N-weakly sequential compact, a contradiction. □

Lemma 1[4,10] *Orlicz function spaces L_M is weakly sequential complete if and only if $M \in \triangle_2$.*

Corollary 1 *In Orlicz function space L_M, all bounded sets are weakly compact if and only if $M \in \triangle_2$ and $N \in \triangle_2$ if and only if L_M is reflexive.*

Theorem 3 *For a set A in l_1. A is relatively weakly compact if and only if A is bounded and A is uniformly summable, i.e., for all $\varepsilon > 0$, there exists an integer I such that for all $A \ni x = x(i), \sum_{i=I+1}^{\infty} |x(i)| < \varepsilon$.*

Proof Necessity. If A is relatively weakly compact, by Hahn-Banach Uniformly Bounded Theorem, A is bounded . Without loss of generality, assume that $A \subset B(l_1)$. By the same argument as in the Theorem 1, we get a contradiction from the Lemma of [8] instead of Lebesgue Theorem of [5].

Sufficiency. When A is bounded and uniformly summable, we shall show that A is relatively compact. For any sequence $\{x_n\} \subset A$, by Diagonal Selection Principle, we have a subsequence, still written as $\{x_n\}$, such that for all i,$x_n(i) \to x(i)$. By Banach-Alaoglu Theorem, we get that $x \in l_1$. Since A is uniformly summable, it follows that for every $\varepsilon > 0$ there is an integer I such that for all $z \in A$, $\sum_{i=I+1}^{\infty} |z(i)| < \varepsilon$, so for all n, $\sum_{i=I+1}^{\infty} |x_n(i)| < \varepsilon$,we get $\sum_{i=I+1}^{\infty} |x(i)| \leq \varepsilon$. From $x_n(i) \to x(i)$, there exists an integer n_0 such that for all $n \geq n_0$,

$$\|x_n - x\|_1 \leq \sum_{i=1}^{I} |x_n(i) - x(i)| + \sum_{i=I+1}^{\infty} |x_n(i)| + \sum_{i=I+1}^{\infty} |x(i)| \leq \varepsilon + \varepsilon + \varepsilon = 3\varepsilon,$$

i.e., $\|x_n - x\|_1 \to 0 \quad (n \to \infty)$. □

Remark In the proof of Theorem 3, we have shown that for $A \subset l_1$, A is relatively compact if and only if A is relatively weakly compact if and only if A is bounded and uniformly summable.

Corollary 2 (Shur's)[6] *In $\{x_n\} \subset l_1$, its weak convergence and norm convergence coincide.*

Theorem 4 *In Orlicz sequence space l_M, all bounded sets are relatively l_N-weakly sequential compact if and only if $N \in \delta_2$.*

Proof Sufficiency. For a bounded set A, as that of Theorem 2, we can assume that $A \subset B(l_M)$, so for all $x \in A$, $\rho_M(x) \leq 1$. By $N \in \delta_2$,i.e., there exist $u_0 > 0$,$1 > \beta > 0$ and $a < 1$ such that for all $|u| \leq u_0, M(\beta u) \leq a\beta M(u)$. Take an integer I with $IM(u_0) \geq 1$, thus

$$\begin{aligned}\lim_{n\to\infty} \sup_{x\in A} \frac{\rho_M(\beta^n x)}{\beta^n} &\leq \lim_{n\to\infty} \sup_{x\in A}\left[\frac{IM(\beta^n u_0 \chi|_x)}{\beta^n} + a^n \rho_M(x)\right] \\ &\leq \lim_{n\to\infty}\left[\frac{IM(\beta^n u_0)}{\beta^n} + a^n\right] \to 0.\end{aligned}$$

By [8], we get that A is relatively l_N-weakly compact.

Necessity. Also as in the proof of Theorem 2, we assume that l_M is equipped with Orlicz norm. Suppose that $N \notin \delta_2$, then there exists $v_n \searrow 0$ such that

$$N((1 + \frac{1}{n})v_n) > 2^{n+1} N(v_n), \quad N(v_n) < \frac{1}{2^n}.$$

Take integers I_n such that

$$I_n N(v_n) \leq \frac{1}{2^n}, \quad (I_n + 1) N(v_n) > \frac{1}{2^n}.$$

Let

$$v(i) = \begin{cases} v_n & \text{if } I_1 + \cdots + I_{n-1} < i \leq I_1 + \cdots + I_n, \ n = 1, 2, \ldots \\ 0 & \text{otherwise.} \end{cases}$$

Then $\rho_N(v) = \sum_{n=1}^{\infty} I_n N(v_n) \le 1$. Let $v_n = v_n \sum_{i=I_1+\ldots+I_{n-1}+1}^{I_1+\ldots+I_n} e_i$, we see that $1 \ge \|v_n\|_{(N)} \ge \frac{1}{1+\frac{1}{n}}$. By [9], there exist $x_n \in L_M$ with $\|x_n\|^0 = 1$, $x_n = u_n \sum_{i=I_1+\ldots+I_{n-1}+1}^{I_1+\ldots+I_n} e_i$ and $\langle v_n, x_n \rangle = \|v_n\|_{(N)}$. Let $A = \{x_n\}$. Then $A \subset B(l_M)$, but A is not l_N-weakly compact. In fact, $\lim_{n\to\infty} \sup_m \langle v|_{i>I_{n-1}}, x_m \rangle \ge \lim_{n\to\infty} \langle v_n, x_n \rangle = \lim_{n\to\infty} \|v_n\|_{(N)} = \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}} = 1$. By [8], A is not relatively l_N-weakly sequential compact, a contradiction $\square$

Lemma 2[2] *In separable Orlicz spaces, $\rho_M(x) \to 0$ if and only if $\|x\| \to 0$.*

Lemma 3[3] *Let X be a Banach lattice. X is sequential weakly completed if and only if X is a KB space.*

Obviously, for the real number relation "$\le$ ", l_M forms an ideal Banach lattice[2].

Theorem 5 *Orlicz sequence space l_M is a KB space if and only if $M \in \delta_2$.*

Proof As in [3], we only need to show

1. $0 \le x_n \downarrow 0 \Rightarrow \|x_n\| \to 0$ if and only if $M \in \delta_2$.
2. $0 \le x_n \uparrow, \sup_n \|x_n\| < \infty \Rightarrow \exists x, \|x_n - x\| \to 0$ if and only if $M \in \delta_2$.

1-1. If $0 \le x_n \downarrow 0 \Rightarrow \|x_n\| \to 0$ then $M \in \delta_2$. Otherwise , by Theorem 4, there exists $u_n \searrow 0$ such that

$$M((1+\frac{1}{n})u_n) > 2^{n+1} M(u_n), \quad M(u_n) < \frac{1}{2^n}.$$

Take integers I_n with $I_n M(u_n) \le \frac{1}{2^n}$, $(I+1)M(u_n) > \frac{1}{2^n}$. Define

$$x = \sum_{n=1}^{\infty} u_n \sum_{i=I_1+\ldots+I_{n-1}+1}^{I_1+\ldots+I_n} e_i,$$

then $\rho_M(x) = \sum_{n=1}^{\infty} I_n M(u_n) \le 1$. Define $x_n = \sum_{j=n}^{\infty} u_j \sum_{i=I_1+\ldots+I_{n-1}+1}^{I_1+\ldots+I_n} e_i$. then $0 \le x = x_1 \ge x_2 \ge x_3 \ge \ldots$ and for all $i, x_n(i) \to 0$ i.e., $x_n \downarrow 0$ but $\|x_n\| \ge \frac{1}{1+\frac{1}{n}} \not\to 0$, a contradiction.

1-2. If $M \in \delta_2$ then $0 \le x_n \downarrow 0 \Rightarrow \|x_n\| \to 0$. For $\varepsilon > 0$, take integer I, $\sum_{i>I} M(x_1(i)) < \varepsilon$. Since $x_n \downarrow 0$, take n_0 such that for all $n \ge n_0$, $\sum_{i=1}^{I} M(x_n(i)) < \varepsilon$,

$$\sum_{i=1}^{\infty} M(x_n(i)) = \sum_{i=1}^{I} M(x_n(i)) + \sum_{i=I+1}^{\infty} M(x_n(i)) \le \varepsilon + \sum_{i=I+1}^{\infty} M(x_1(i)) \le 2\varepsilon.$$

By Lemma 2, $\|x_n\| \to 0$.

2-1. If $0 \le x_n \uparrow, \sup_n \|x_n\| < \infty \Rightarrow \exists x, \|x_n - x\| \to 0$ then $M \in \delta_2$. Otherwise there exist x and x_n as in the proof of 1-1. Define $y_n = x - x_{n-1}$, thus $0 \le y_n \uparrow x$, $\sup_n \|x_n\| \le \|x\| \le 1$ but $\|y_n - x\| \ge \|x_n\| \ge \frac{1}{1+\frac{1}{n}} \not\to 0$, a contradiction.

2-2. If $M \in \delta_2$, then $0 \le x_n \uparrow$, $\sup_n \|x_n\| < \infty \Rightarrow \exists x, \|x_n - x\| \to 0$. Without loss of generality, we assume that $\|x_n\| \le 1$. For each i, let $x(i) = \sup_n x_n(i)$. then for all integers I, $\sum_{i=1}^{I} M(x(i)) \le \sup_n \sum_{i=1}^{I} M(x_n(i)) \le \sup_n \rho_M(x_n) \le 1$, so $\rho_M(x) = \sum_{i=1}^{\infty} M(x(i)) \le$

1. On the other hand, for any $\varepsilon > 0$ take an integer I such that $\sum_{i=1}^{I} M(x(i)) < \varepsilon$. By $x_n(i) \to x(i)$, there exists n_0 such that for all $n \geq n_0$, $\sum_{i=1}^{I} M(x_n(i) - x(i)) < \varepsilon$. Then

$$\sum_{i=1}^{\infty} M(x_n(i) - x(i)) = \sum_{i=1}^{I} M(x_n(i) - x(i)) + \sum_{i=I+1}^{\infty} M(x_n(i) - x(i))$$
$$\leq \varepsilon + \frac{K}{2} \sum_{i=I+1}^{\infty} [M(x_1(i)) + M(x_1(i))] \leq \varepsilon + \frac{K}{2}(\varepsilon + \varepsilon) \leq (2 + K)\varepsilon,$$

by Lemma 2, $\|x_n\| \to 0, (n \to \infty)$. □

Corollary 3 *Orlicz sequence space l_M is sequential weakly completed iff $M \in \delta_2$.*

Corollary 4 *In Orlicz sequence spaces l_M, all bounded sets are relatively weakly compact if and only if $M \in \delta_2$ and $N \in \delta_2$ if and only if l_M is reflexive.*

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弱紧集与 Orlicz 空间

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摘 要: 本文圆满完善了 [1] 的结果，并由此给出了 Orlicz 空间自反性与 Shur 定理的新证明。