Journal of Mathematical Research & Exposition
Vol.21, No.4, 543-546, November, 2001

The Argument Distribution of Infinite Order Meromorphic Functions *

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Abstract: We study the argument distribution of infinite order meromorphic functions and obtain distribution theorem which combines the infinite order meromorphic functions with its derived function.

Key words: meromorphic function; infinite order.

Classification: AMS(1991) 30C45/CLC O174.51

Document code: A **Article ID**: 1000-341X(2001)04-0543-04

We use $n(D,f=\alpha), n(r,f=\alpha)$ and $n(r,\theta,\varepsilon,f=\alpha)$ to denote respectively the number of zero point of the function $f(z)-\alpha$ on the region $D, \{z||z|<r\}$ and $\{z||z|\le r$ and $|\arg z-\theta|\le\varepsilon\}$, where repeated zero point will be counted according to repeated numbers.

Definition *Let $f(z)$ be a non-constant meromorphic function on the region D and $m>4$. Circle C in D is called full circle of the function $f(z)$ with index number m, if $n(c,f=\alpha)\ge m$ for any number α, at most except some α, may be limited within two spherical circular of the radius being e^{-m}.*

Lemma 1[1] *If $f(z)$ is an infinite order meromorphic function, then there exists a positive number $r_1 (r_1>1)$ which depends only on the function $f(z)$ and the type function $u(r)$.*

If $r\in s_1, r>r_1, q>1$ and

$$m=\frac{T(r,f)}{6\beta q^2\log r(\frac{4}{5}\log r+1)}>4,$$

then there exists a point z_0 in circular ring $r^{1/5}<|z|<r$, and circle $|z-z_0|<\frac{1}{q}|z_0|$ is a full circle of the function $f(z)$ with index number m, where β is a positive number.

$$S_1=\{r|r\ge r_0, T(r,f)>[u(r)]^{\frac{6}{7}}\}.$$

*Received date: 1998-08-04
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Lemma 2[2] *Let $f(z)$ be a meromorphic function in $|z|<1$. If $n(1,f=0)+n(1,f=\infty)+n(1,f^{(k)}=1)<N$, then for any complex number a we have $n(\frac{1}{32},f=a)<C_k\{N+\log\frac{1}{|f(z^*),a|}+\log^+\log^+|f(z^*)|\}$, where $|f(z^*),a|$ is spherical distance of the point $f(z^*)$ from a, $z^*\in\{z||z|<\frac{1}{32}\}$, C_k is a constant depending on K and*

$$\prod_{q=1}^{n(1,f=\infty)}|z^*-\beta_q|>(\frac{1}{400e})^{n(1,f=\infty)},$$

where $\beta_q(q=1,2,\cdots,n(1,f=\infty))$ is the extreme point of the function $f(z)$ in $|z|<1$.

Theorem 1 *Let $f(z)$ be an infinite order holomorphic function. If its type function is $u(r)=r^{p(r)}$, then there exists a half straight line $B:\arg z=\theta_0(0\le\theta_0<2\pi)$, possessing the following property: If K is any positive integer, α,β are any two finite complex numbers and $\beta\ne 0$, then for any $\varepsilon\in(0,\frac{\pi}{2})$, there hold*

$$\overline{\lim_{r\to\infty}}\log\{n(r,\theta_0,\varepsilon,f=\alpha)+n(r,\theta_0,\varepsilon,f^{(k)}=\beta)\}/[p(r)\log r]=1$$

Proof Evidently there exist a sequence $\{r^{(n)}\}$ satisfying following conditions:

$$r^{(n)}\in S_1,r^{(n)}>r_1(n=1,2,3,\cdots)$$

$$\lim_{n\to\infty}r^{(n)}=+\infty,\ \lim_{n\to\infty}\frac{\log T(r^{(n)},f)}{\log u(r^{(n)})}=1.$$

Take $r=r^{(n)},q=64\log u(r^{(n)})$ we have

$$m=m_n=\frac{T(r,f)}{6\beta q^2\log r(\frac{4}{5}\log r+1)}>4,\ n=1,2,3,\cdots$$

By Lemma 1, for any positive integer n, there exists a point $z_n=|z_n|e^{i\theta_n}(0\le\theta_n<2\pi)$ on the circular ring $(r^{(n)})^{1/5}<|z|<|r^{(n)}|$ and make $T_n:|z-z_n|<\frac{|z_n|}{64\log u(r^{(n)})}$ be a full circle of the function $f(z)$ with index number m_n.

We may assume the sequence $\{\theta_n\}$ has a limit θ_0 when $n\to\infty$,otherwise, we may take a convergence subsequence of $\{\theta_n\}$. Now we shall prove that θ_0 possesses the property of the theorem.

Otherwise, there exist a positive integer K_0, two complex numbers $\alpha_0,\beta_0(\beta_0\ne 0)$ and a positive number $\varepsilon_0\in(0,\frac{\pi}{2})$. When r is sufficiently large

$$n(r,\theta_0,\varepsilon_0,f=\alpha_0)+n(r,\theta_0,\varepsilon_0,f^{(k)}=\beta_0)<[u(r)]^{1-\eta},$$

where η is a positive number.

Let $g(z)=\frac{f(z)-\alpha_0}{\beta_0}$. Then Γ_n is also a full circle of the function $g(z)$ with index number m_n. That means for any complex number α, we have $n(\Gamma_n,f=\alpha)\ge m_n$, at most except some α limited within two spherical circular S'_n,S''_n of the radius being e^{-m}

Let $\varepsilon_j=\frac{1}{64\log u(r^{(j)})},\Gamma'_j:|z-z_j|<32\varepsilon_j|z_j|$. We can easily prove when j is sufficiently large, Γ'_j is contained within angular domain $|\arg z-\theta_0|<\varepsilon_0$.

When j is sufficiently large, for any fixed j, let $h_j(t) = \frac{g(z_j+32\varepsilon_j|z_j|t)}{(32\varepsilon_j|z_j|)^{k_0}}$ then $h_j(t)$ is a holomorphic function on $|t| \le 1$ and

$$\begin{aligned} &n(1, h_j = 0) + n(1, h_j^{(k_0)} = 1) = n(\Gamma_j', g = 0) + n(\Gamma_j', g^{(k_0)} = 1) \\ &\quad \le n(|z_j| + 32\varepsilon_j|z_j|, \theta_0, \varepsilon_0, f = \alpha_0) + n(|z_j| + 32\varepsilon_j|z_j|, \theta_0, \varepsilon_0, f^{(k_0)} = \beta_0) \\ &\quad < [u(|z_j| + 32\varepsilon_j|z_j|)]^{1-\eta}. \end{aligned}$$

By Lemma 2, for any complex number a, there is

$$\begin{aligned} &n(\frac{1}{32}, h_j = \frac{a}{(32\varepsilon_j|z_j|)^{k_0}}) = n(\Gamma_j, g = a) \\ &\quad < C_{k_0}\{[u(|z_j| + 32\varepsilon_j|z_j|)]^{(1-\eta)} + \log \frac{1}{|g(z_j^*)/(32\varepsilon_j|z_j|)^{k_0}, \frac{a}{(32\varepsilon_j|z_j|)^{k_0}}|} + \\ &\quad \log^+ \log^+ |\frac{g(z_j^*)}{(32\varepsilon_j|z_j|)^{k_0}}|\} \end{aligned}$$

Since $g(z)$ is a holomorphic function, hence when $0 < r < \rho < +\infty$

$$T(r, g) \le \log^+ M(r, g) \le \frac{\rho + r}{\rho - r} T(\rho, g)$$

by $z_j^* \in \Gamma_j$, we have

$$\log^+ |g(z_j^*)| \le \frac{|z_j| + 2\varepsilon_j|z_j| + |z_j^*|}{|z_j| + 2\varepsilon_j|z_j| - |z_j^*|} T(|z_j| + 2\varepsilon_j|z_j|, g) \le \frac{3|z_j|}{\varepsilon_j|z_j|} T(|z_j| + 2\varepsilon_j|z_j|, g),$$

combining $\lim_{r\to\infty} \frac{\log u(R)}{\log u(r)} = 1, \lim_{r\to\infty} \frac{\log T(r,g)}{\log u(r)} = 1$. We have

$$\log^+ \log^+ |g(z_j^*)| \le C_1 \log T(|z_j| + 2\varepsilon_1|z_j|, g) \le C \log u(|z_j|) \le C \log U(r^{(j)})$$

where C_1 and C are positive constants.

When α does not belong to S_j', S_j'' and $S_j''' : |g(z_j^*), \alpha| \le 1, m_j \le n(\Gamma_j, g = \alpha) \le 2C_{k_0}[u(|z_j| + 64\varepsilon_j|z_j|)]^{1-\frac{\eta}{2}}$. Hence we have

$$\lim_{j\to\infty} \frac{\log m_j}{\log u(r^{(j)})} \le \lim_{j\to\infty} \frac{(1 - \frac{\eta}{2}) \log 2C_{k_0}[U(|z_j| + 64\varepsilon_j|z_j|)]}{\log u(r^{(j)})} \le (1 - \frac{\eta}{2}).$$

This is a contradiction to the fact $\lim_{j\to\infty} \frac{\log m_j}{\log u(r^{(j)})} = 1$. Hence the proof of Theorem 1 is completed.

Theorem 2 *Let $f(z)$ be an infinite order meromorphic function on the open plane, then there exists a half straight line $\arg z = \theta_0 (0 \le \theta_0 < 2\pi)$, possessing following property:*

If K is any positive integer, α, β are any two finite complex numbers and $\beta \ne 0$, then for any $\varepsilon \in (0, \frac{\pi}{2})$ we have

$$\overline{\lim_{r\to\infty}} \frac{\log\{n(r, \theta_0, \varepsilon, f = \infty) + n(r, \theta_0, \varepsilon, f = \alpha) + n(r, \theta_0, \varepsilon, f^{(k)} = \beta)}{\rho(r) \log r} = 1.$$

The first part of proof of Theorem 2 can be completed in a similar way to that of Theorem 1. In order to prove $\log^+\log^+|g(z_j^*)| < C\log u(|z_j|)$, note that from by Poisson-Jensen formula, we have

$$\log|g(z_j^*)| \leq \frac{3|z_j|+2|z_j|}{3|z_j|-2|z_j|}m(3|z_j|,g) + \sum_{|b'_{j_q}|\leq|3z_j|}\log|\frac{(3|z_j|)^2 - b'_{j_q}z_j^*}{3|z_j|(z_j^* - b'_{j_q})}|$$

$$\leq 5m(3|z_j|,g) + n(3|z_j|,g=\infty)\log 6|z_j| + \sum_{|b'_{j_q}|\leq 3|z_j|}\log\frac{1}{|z_j^* - b'_{j_q}|},$$

where b'_{j_q} are the extrem points of the function $g(z)$ in $|z|\leq 3|z_j|$.

We can suppose when j is sufficiently large, $\varepsilon_j|z_j|\geq 1$, otherwise, we may consider

$$\Gamma_n^* : |z - z_n| \leq \varepsilon_n^*|z_n| = 1.$$

When $|z_j^* - b'_{j_q}| \geq 32\varepsilon_j|z_j|, \log\frac{1}{|z_j^*-b'_{j_q}|} < 0$, hence we have

$$\sum_{|b'_{j_q}|\leq 3|z_j|}\log\frac{1}{|z_j^* - b'_{j_q}|} \leq \sum_{|z_j^*-b'_j|\leq 32|z_j|}\log\frac{1}{|z_j^* - b'_{j_q}|}$$

$$= \sum_{|t_j^*-b_{j_q}|<1}\log\frac{1}{|z_j + 32\varepsilon_j|z_j|t_j^* - (z_j + 32\varepsilon_j|z_j|b_{j_q})}$$

$$< n(1,h_j=\infty)\log(400e) = n(\Gamma_j,g=\infty)\log(400e).$$

Because $g(z)$ is an infinite order meromorphic function, we get $\log^+\log^+|g(z^*)| < 2\log u(|z_j|)$.

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无限级亚纯函数的幅角分布

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摘 要: 本文研究了无限级亚纯函数的幅角分布，将有限级亚纯函数的两个分布定理推广到无限级亚纯函数中.