Journal of Mathematical Research & Exposition
Vol.21, No.4, 547-551, November, 2001

# The Decomposition of a Kind of Complex Matrices *

*LI Yang-ming*[1,2]

(1. Dept. of Math., Guangdong Educational College, Guangzhou 510303, China;
2. Dept. of Math., Zhongshan University, Guangzhou 510275, China)

**Abstract**: We give an affirmative answer to the open problem proposed by Dr. Fuzhen Zhang in the paper "Quaternions and matrices of quaternions"( LAA251:21-57 ) ,that is, there exists a $2n \times 2n$ complex matrix $B$ such that $\begin{pmatrix} A_1 & A_2 \\ -\overline{A_2} & \overline{A_1} \end{pmatrix} = \overline{B}B$, for any $n \times n$ complex matrices $A_1, A_2$.

**Key words**: open problem; Jordan block, companion matrix of $p(\lambda)$.

**Classification**: AMS(1991) 15A33/CLC O151.21

**Document code**: A **Article ID**: 1000-341X(2001)04-0547-05

## 1. Introduction

As usual, let $C$ and $R$ denote the fields of the complex and real numbers respectively, $M_n(C)$ and $M_n(R)$ denote the collection of all $n$-by-$n$ matrices with complex and real number entries respectively.

In recent years, the matrices with quaternion entries have gained much attention in matrix theory. One of the effective approaches to studying matrices of quaternion may be converting a quaternion matrix to a pair of complex matrices: writing quaternion matrix $A = A_1 + A_2 j \mapsto \begin{pmatrix} A_1 & A_2 \\ -\overline{A_2} & \overline{A_1} \end{pmatrix}$,where $A_1, A_2$ are complex matrices. The isomorphism was first introduced by Wolf (1936,AMS Bulletin), later used by Lee (1949,Proc.R.I.A.), Huang, So, Thompson, Zhang, $\cdots$. So the properties of the complex matrices having following form:

$$\begin{pmatrix} A_1 & A_2 \\ -\overline{A_2} & \overline{A_1} \end{pmatrix}$$

where $A_1, A_2 \in M_n(C)$, are very interesting. Dr. Zhang proposed the following question (QUESTION 5.3 in [1])

*Received date: 1998-11-19
**Biography**: LI Yang-ming (1965- ), male, born in Dongxiang city, Jiangxi Province, currently an associate professor of ECG, Ph.D. student of Zhongshan University.
**E-mail**: liyangming@gdei.edu.cn

**Question** *Does there exist a matrix $B$ such that* $\begin{pmatrix} A_1 & A_2 \\ -\overline{A_2} & \overline{A_1} \end{pmatrix} = \overline{B}B$, *for any* $A_1, A_2 \in M_n(C)$ ?

In this paper, we give an affirm answer to the problem, that is,

**Theorem** *There does exist a matrix* $B \in M_{2n}(C)$ *such that* $\begin{pmatrix} A_1 & A_2 \\ -\overline{A_2} & \overline{A_1} \end{pmatrix} = \overline{B}B$ *for any* $A_1, A_2 \in M_n(C)$.

Let $J_k(\lambda_0)$ denote the $k \times k$ Jordan block as follows:

$$\begin{pmatrix} \lambda_0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & \lambda_0 & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & & & & \\ 0 & 0 & \cdots & 1 & \lambda_0 & 0 \\ 0 & 0 & \cdots & 0 & 1 & \lambda_0 \end{pmatrix}$$

Let

$$N = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & -a_n \\ 1 & 0 & 0 & \cdots & 1 & -a_{n-1} \\ 0 & 1 & 0 & \cdots & 0 & -a_{n-2} \\ \cdots & \cdots & & & & \\ 0 & 0 & 0 & \cdots & 1 & -a_1 \end{pmatrix},$$

where $a_i \in R, i = 1, 2, \cdots, n$.Then the non-constant invariant factor of $N$ is $p(\lambda) = \lambda^n + a_1\lambda^{n-1} + \cdots + a_{n-1}\lambda + a_n$, $N$ is so-called companion matrix([3]) of $p(\lambda)$.

If $\lambda_0$ is a non-real complex number, $\overline{\lambda}_0$ is the conjugate of $\lambda_0$, then the non-constant invariant factor of complex matrix $\begin{pmatrix} J_k(\lambda_0) & \\ & J_k(\overline{\lambda}_0) \end{pmatrix}$ is $f(\lambda) = (\lambda - \lambda_0)^k(\lambda - \overline{\lambda}_0)^k = [\lambda^2 - (\lambda_0 + \overline{\lambda}_0)\lambda + \lambda_0\overline{\lambda}_0]^k$, obviously the coefficients of $f(\lambda)$ are real, denote its companion matrix as $N_k(\lambda_0)$, then $N_k(\lambda_0) \in M_{2k}(R)$ and $\begin{pmatrix} J_k(\lambda_0) & \\ & J_k(\overline{\lambda}_0) \end{pmatrix}$ is similar to $N_k(\lambda_0)$ in $C$, i.e., $\begin{pmatrix} J_k(\lambda_0) & \\ & J_k(\overline{\lambda}_0) \end{pmatrix} \sim N_k(\lambda_0)$.

## 2. Some lemmas

**Lemma 1** *Let $A \in M_n(C)$. If $A$ is similar to a square of a real matrix, then there exists $B \in M_n(C)$ such that $A = \overline{B}B$.*

**Proof** Suppose $A = P^{-1}D^2P$,$P$ is an invertible complex matrix and $D \in M_n(R)$. Then $A = P^{-1}D\overline{P} \cdot (\overline{P})^{-1}DP = \overline{(\overline{P})^{-1}DP} \cdot (\overline{P})^{-1}DP$. Take $B = (\overline{P})^{-1}DP$, then we get the result.

**Lemma 2** (Corollary 6.3 of [1]) *For any $n \times n$ complex matrices $A$ and $B$, the block matrix*

$$\begin{pmatrix} A & B \\ -\overline{B} & \overline{A} \end{pmatrix}$$

*has the Jordan canonical form*

$$\begin{pmatrix} J & 0 \\ 0 & \bar{J} \end{pmatrix},$$

*where $J$ is a Jordan form of some $n \times n$ complex matrix. Consequently, all the Jordan blocks are paired.*

**Lemma 3** *Let $\lambda_0 \in C$.*

*(1) $\lambda_0 \neq 0$,then $J_k(\lambda_0) \sim J_k^2(\sqrt{\lambda_0}) \sim J_k^2(-\sqrt{\lambda_0})$;*

*(2) $\lambda_0 = 0$,then $\begin{pmatrix} J_k(0) & \\ & J_k(0) \end{pmatrix} \sim J_{2k}^2(0)$;*

*(3) $\lambda_0$ is a non-real complex number, then $N_k(\lambda_0) \sim N_k^2(\sqrt{\lambda_0})$.*

**Proof** (1) Since

$$J_k^2(\sqrt{\lambda_0}) = \begin{pmatrix} \lambda_0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 2\sqrt{\lambda_0} & \lambda_0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 2\sqrt{\lambda_0} & \lambda_0 & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & & & & & \\ 0 & 0 & 0 & \cdots & 2\sqrt{\lambda_0} & \lambda_0 & 0 \\ 0 & 0 & 0 & \cdots & 1 & 2\sqrt{\lambda_0} & \lambda_0 \end{pmatrix},$$

then its characteristic matrix is

$$\begin{pmatrix} \lambda - \lambda_0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ -2\sqrt{\lambda_0} & \lambda - \lambda_0 & 0 & \cdots & 0 & 0 & 0 \\ -1 & -2\sqrt{\lambda_0} & \lambda - \lambda_0 & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & & & & & \\ 0 & 0 & 0 & \cdots & -2\sqrt{\lambda_0} & \lambda - \lambda_0 & 0 \\ 0 & 0 & 0 & \cdots & -1 & -2\sqrt{\lambda_0} & \lambda - \lambda_0 \end{pmatrix}.$$

So its $k$-by-$k$ determinant factor is $(\lambda - \lambda_0)^k$ . Noticing that the $(k-1)$-by-$(k-1)$ minor in $2nd, 3rd, \cdots, kth$ rows and $1st, 2nd, \cdots, (k-1)th$ columms is

$$g(\lambda) = \begin{vmatrix} -2\sqrt{\lambda_0} & \lambda - \lambda_0 & 0 & \cdots & 0 & 0 \\ -1 & -2\sqrt{\lambda_0} & \lambda - \lambda_0 & \cdots & 0 & 0 \\ \cdots & \cdots & & & & \\ 0 & 0 & 0 & \cdots & -2\sqrt{\lambda_0} & \lambda - \lambda_0 \\ 0 & 0 & 0 & \cdots & -1 & -2\sqrt{\lambda_0} \end{vmatrix}.$$

Since $g(\lambda_0) = (-2\sqrt{\lambda_0})^{k-1} \neq 0$,so $g(\lambda)$ does not divide by $\lambda - \lambda_0$,thus the greatest common factor of $g(\lambda)$ and $(\lambda - \lambda_0)^k$ is 1.Since the (k-1)-by-(k-1) determinant factor is the divisor of the k-by-k determinant factor and every (k-1)-by-(k-1) minor, so the (k-1)-by-(k-1) determinant factor is 1,thus the elementary factor of $J_k^2(\sqrt{\lambda_0})$ is $(\lambda - \lambda_0)^k$, the same as $J_k(\lambda_0)$.

By the same way, we get that the elementary factor of $J_k^2(-\sqrt{\lambda_0})$ as $(\lambda - \lambda_0)^k$, so (1) holds.

(2) Since

$$J_k^2(0) = \begin{pmatrix} 0 & & & & \\ 0 & \ddots & & & \\ 1 & \ddots & \ddots & & \\ & \ddots & \ddots & & \\ & & 1 & 0 & 0 \end{pmatrix},$$

by the same way as in (1), we can get the elementary factors of $J_{2k}^2(0)$ as $\lambda^k, \lambda^k$. Thus (2) holds.

(3) Since $N_k(\sqrt{\lambda_0}) \sim \begin{pmatrix} J_k(\sqrt{\lambda_0}) & \\ & J_k(\overline{\sqrt{\lambda_0}}) \end{pmatrix}$, so

$$N_k^2(\sqrt{\lambda_0}) \sim \begin{pmatrix} J_k^2(\sqrt{\lambda_0}) & \\ & J_k^2(\overline{\sqrt{\lambda_0}}) \end{pmatrix} \sim \begin{pmatrix} J_k(\lambda_0) & \\ & J_k(\overline{\lambda_0}) \end{pmatrix} \sim N_k(\lambda_0).$$

## 3. The proof of theorem

By lemma 2, $\begin{pmatrix} A_1 & A_2 \\ -\overline{A_2} & \overline{A_1} \end{pmatrix} \sim \begin{pmatrix} J & 0 \\ 0 & \overline{J} \end{pmatrix}$. Suppose

$$J = \begin{pmatrix} J_{n_1}(\lambda_1) & & & \\ & J_{n_2}(\lambda_2) & & \\ & & \ddots & \\ & & & J_{n_k}(\lambda_k) \end{pmatrix}, n_1 + n_2 + \cdots + n_k = n,$$

then $\begin{pmatrix} A_1 & A_2 \\ -\overline{A_2} & \overline{A_1} \end{pmatrix}$ is permutation similar to

$$\begin{pmatrix} J_{n_1}(\lambda_1) & & & & & & \\ & J_{n_1}(\overline{\lambda_1}) & & & & & \\ & & J_{n_2}(\lambda_2) & & & & \\ & & & J_{n_2}(\overline{\lambda_2}) & & & \\ & & & & \ddots & & \\ & & & & & J_{n_k}(\lambda_k) & \\ & & & & & & J_{n_k}(\overline{\lambda_k}) \end{pmatrix},$$

If $\lambda_i$ is a positive real number, then $\sqrt{\lambda_i} \in R$, by Lemma 3(1),we get

$$\begin{pmatrix} J_{n_i}(\lambda_i) & \\ & J_{n_i}(\overline{\lambda_i}) \end{pmatrix} \sim \begin{pmatrix} J_{n_i}^2(\sqrt{\lambda_i}) & \\ & J_{n_i}^2(\sqrt{\lambda_i}) \end{pmatrix} = \begin{pmatrix} J_{n_i}(\sqrt{\lambda_i}) & \\ & J_{n_i}(\sqrt{\lambda_i}) \end{pmatrix}^2,$$

the matrix in the right side is a square of a real matrix;

If $\lambda_i = 0$, then by lemma 3(2),we have

$$\begin{pmatrix} J_{n_i}(\lambda_i) & \\ & J_{n_i}(\overline{\lambda_i}) \end{pmatrix} = \begin{pmatrix} J_{n_i}(0) & \\ & J_{n_i}(0) \end{pmatrix} \sim J^2_{2n_i}(0),$$

the matrix in the right side is also a square of a real matrix;

If $\lambda_i$ is a negative real number, suppose $\lambda_i = -a_i^2, a_i^2 \in R^+$, then $\sqrt{\lambda_i} = a_i i, \overline{\sqrt{\lambda_i}} = -a_i i$. So by lemma 3(1),

$$\begin{pmatrix} J_{n_i}(\lambda_i) & \\ & J_{n_i}(\overline{\lambda_i}) \end{pmatrix} = \begin{pmatrix} J_{n_i}(-a_i^2) & \\ & J_{n_i}(-a_i^2) \end{pmatrix}$$

$$\sim \begin{pmatrix} J^2_{n_i}(a_i i) & \\ & J^2_{n_i}(-a_i i) \end{pmatrix} = \begin{pmatrix} J_{n_i}(a_i i) & \\ & J_{n_i}(\overline{a_i i}) \end{pmatrix}^2 \sim N^2_{n_i}(a_i i)$$

the matrix in the right side is a square of a real matrix.

If $\lambda_i$ is a non-real complex number, by lemma 3(3), we have

$$\begin{pmatrix} J_{n_i}(\lambda_i) & \\ & J_{n_i}(\overline{\lambda_i}) \end{pmatrix} \sim N_{n_i}(\lambda_i) \sim N^2_{n_i}(\sqrt{\lambda_i}).$$

the matrix in the right side is also a square of a real matrix.

We have proved that every paired Jordan block $\begin{pmatrix} J_{n_i}(\lambda_i) & \\ & J_{n_i}(\overline{\lambda_i}) \end{pmatrix}$ is similar to a square of real matrix, then so does $\begin{pmatrix} A_1 & A_2 \\ -\overline{A_2} & \overline{A_1} \end{pmatrix}$. By lemma 1, the Theorem holds.

## References:


[1] ZHANG Fu-zhen. *Quaternions and matrices of quaternions* [J]. Linear Algebra and Its Applications, 1997, **251**: 21–57.

[2] HORN R A, JOHNSON C R. *Matrix Analysis* [M]. Cambridge U.P., 1985.

[3] JACOBSON N. *Basic Algebra I* [M]. San Francisco:W H Freeman and Company, 1974, 191.


# 一类复矩阵的分解性


李 样 明 [1,2]

(1. 广东教育学院数学系, 广东 广州 510303; 2. 中山大学数学系, 广东 广州 510275)



**摘 要**: 给出了文 [1] 中一个公开问题的肯定回答, 即对任意的 $n \times n$ 复矩阵 $A_1, A_2$ 存在 $2n \times 2n$ 复矩阵 $B$, 使得 $\begin{pmatrix} A_1 & A_2 \\ -\overline{A_2} & \overline{A_1} \end{pmatrix} = \overline{B}B$.