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Central Circuit Coverings of Octahedrites and Medial Polyhedra *

Michel Deza[1], *Tayuan Huang*[2], *Ko-Wei Lih*[3]

(1. CNRS and Ecole Normale Supérieure, Paris, France

2. Dept. of Appl. Math., National Chiao-Tung University Hsin-Chu, Tianwan;

3. Inst. of Math., Academia Sinica Nankang, Taipei, Taiwan)

Abstract: An octahedrite is a 4-valent polyhedron with only 3-faces and 4-faces. We study edge-partitions of some octahedrites, medial and related polyhedra into central circuits.

Key words: polyhedra; plane graphs; central circuits; alternating links.

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1. Introduction

A graph G consists of a vertex set $V(G)$ and an edge set $E(G)$ such that each edge is assigned either one or two vertices as its ends. A graph is said to be *simple* if no edge has only one end vertex and no two edges have identical end vertices. We only deal with graphs whose vertex and edge sets are finite. A *plane* graph is a particular drawing of a graph in the Euclidean plane using smooth curves that cross each other only at the vertices of the graph. A graph that has at least one such drawings is called *planar*.

A *walk* of length ℓ in a graph G is a sequence $v_0e_1v_1\cdots e_\ell v_\ell$, where $v_0, v_1, \ldots, v_\ell$ are vertices of G, $e_1, e_2, \ldots, e_\ell$ are edges of G, and v_{i-1} and v_i are the ends of e_i for $1 \le i \le \ell$. A *circuit* is a walk without repeated edges such that $v_0 = v_\ell$. A *cycle* is a circuit without repeated vertices.

By a polyhedron, we mean a convex 3-dimensional polytope P. The vertices and the edges of P form a simple, plane, and 3-connected graph. We use P to denote both the polyhedron and its graph (the skeleton). A polyhedron P is called *semiregular* if the faces of P are regular and the group of symmetries of P acts transitively on the vertices of P. The semiregular polyhedra include the 5 well-known Platonic solids, 13 Archimedean solids, and two infinite families of prisms and antiprisms. Let $u_1, u_2, \ldots, u_m$ and $v_1, v_2, \ldots, v_m$ be two concentric cycles in the plane. A *prism* $Prism_m$ is formed when every u_i is joined

*Received date: 2001-07-06
E-mail: makwlih@sinica.edu.tw

to v_i by an edge. On the other hand, an *antiprism* $APrism_m$ is formed when the cycle $u_1, v_2, u_2, v_3, \ldots, v_m, u_m, v_1, u_1$ is added.

The v-vector $(\ldots, v_i, \ldots)$ of a polyhedron enumerates the numbers v_i's of vertices of degree i. A polyhedron is said to be *simple* when every entry of its v-vector is equal to 3. The p-vector $(\ldots, p_i, \ldots)$ of a polyhedron enumerates the numbers p_i's of faces having i sides. A face having i sides is also called an i-face.

The main objects studied in this paper are so-called *octahedrites*, i.e., 4-valent polyhedra with only 3-faces and 4-faces. The edge set of an octahedron can be partitioned into *central circuits*. We are interested in the distribution of lengths and frequencies of intersection of these circuits. We also study the effects on octahedrites when central circuits are added or removed from them.

Closely related to octahedrites are families of iterated elongations of prisms, antiprisms, pyramids, and bipyramids. We study their edge-partitions into central circuits and their configurations for 3-faces.

Central circuits can be defined for any plane drawing of an Eulerian graph, not necessarily planar. So-called Petri circuits that traverse in "left-right" manner can also defined. In the final section, Petri circuits for fullerenes, i.e., 3-valent polyhedra with only 5-faces and 6-faces, are discussed. A question of Grünbaum is answered; namely, the combinatorial type of a simple 3-polytope is *not* determined by its vector enumerating the numbers of i-faces and its vector enumerating the numbers of Petri circuits of length i.

2. Central circuits

The following is a systematic way of drawing connected 4-regular plane graphs, *C4P* graphs for short. We start out by drawing a closed smooth curve. Each time the curve crosses itself, the point of intersection is regarded as a vertex of the graph to be constructed. In the subsequent stages, closed smooth curves are laid out one by one so that the tracing of a new curve satisfies the following conditions.

(i) It does not pass through any existent vertex of the graph.

(ii) It is required to cross a curve once it meets that curve and every crossing creates a new vertex of the graph.

(iii) It must cross one of the curves that have already been laid out.

After finitely many stages, a C4P graph is constructed.

The above procedure can be reversed such that any C4P graph is decomposable into the constituent circuits. For this purpose, we introduce the notion of a central circuit. Let G be a C4P graph. Suppose that the vertex v is an end of a non-loop edge e. Then rotating e clockwise around v, we successively encounter three adjacent edges of e that are called, respectively, the *left*, the *opposite*, and the *right* neighbor of e. A walk C of G is called a *Petri walk* (cf. [6], page 258) if in tracing the walk we alternately select as next edge the left neighbor and the right neighbor. A circuit C of G is called a *central circuit* if every edge of C is the opposite neighbor of its preceding edge. Once an edge of G is chosen, the tracing of successive opposite edges are uniquely determined. We also note that the construction of the previous paragraph is a successive addition of central circuits. Consequently, we have the following.

Fact

The edge set of a C4P graph G can be decomposed into central circuits in a unique way. Furthermore, if we construct a C4P graph G by successively adding central circuits, then the decomposition of G into central circuits will produce exactly the original circuits.

Central circuits were previously known as *geodesics* ([7]), *straight ahead* ([10]), or *transverse* ([5]) circuits. An edge partition described in the above theorem is also called a *CC-partition* for short. Actually, the number of central circuits in a CC-partition is independent of the plane embedding of a connected 4-regular planar graph ([12]). For a C4P graph G of order n, its *CC-vector* $CC(G) = (\dots, a_i^{\alpha_i}, \dots; \dots, b_j^{\beta_j}, \dots)$ is such that $\dots, a_i, \dots$ and $\dots, b_j, \dots$ are increasing sequences of lengths of all its central circuits, without and with self-intersection, respectively, and α_i and β_j are their multiplicities. Clearly, $\sum_i a_i\alpha_i + \sum_j b_j\beta_j = 2n$. Let $r = r(G)$ denote $\sum_i \alpha_i + \sum_j \beta_j$, i.e., the total number of central circuits. Each of a_i and b_j is even if $r \geq 2$, because any two different circuits intersect even number of times; if $r = 1$, then unique circuit has length $2n$.

For a central circuit C, its *intersection vector* $\mathrm{Int}(C) := (c_0; \dots, c_k^{\gamma_k}, \dots)$ is such that c_0 is the number of self-intersections of C and $(\dots, c_k, \dots)$ is a decreasing sequence of sizes of its intersection with the other $r-1$ circuits, with γ_k denoting respective multiplicities.

We call a C4P graph G *balanced* if all its central circuits of the same length have identical intersection vectors. Any oc_n with $n \leq 21$ is balanced. But Nr. 22-1 with CC-vector $(8^4, 12)$ is not balanced: two its disjoint 8-cycles have intersection vector $(0; 4, 2^2, 0)$ (4 common vertices with the 12-cycle), while two remaining 8-cycles have it $(0; 2^4)$.

For a balanced G, we call it *equilinked* if all its central circuits have the same length.

A C4P graph is called *pure* if its CC-partition consists of only cycles. For example, the graph of an octahedron. A C4P graph is called a *Gaussian* graph if its CC-partition consists of only one circuit. Clearly, this unique central circuit is an Eulerian trail, i.e., a closed walk containing every edge exactly once. Note that a C4P graph may admit Eulerian trails that are not CC-partitions. The smallest example of a Gaussian graph is the antiprism $APrism_4$.

A link diagram is called *alternating* if an over-crossing and an under-crossing appear alternately as the arcs of the link are traversed. When we trace a central circuit of a C4P graph, we alternately designate each passing of vertices as "over" and "under". It can be proved ([14]) that this gives rise to an alternating link. Hence Gaussian graphs will also be called *knot* graphs.

3. Medial graphs

For a connected plane graph G, we denote its planar dual graph by G^*. The *medial* graph $Med(G)$ of G is defined as follows. Place a vertex on every edge of G. Join two new vertices representing edges e_1 and e_2 by an edge if they are incident in G and they lie on the boundary of the same face of G. Two parallel edges of G determine two parallel edges in $\mathrm{Med}(G)$. The medial graph of G is a subgraph of its line graph $L(G)$. For a 3-regular plane graph, they are the same. We also see that $\mathrm{Med}(G) = \mathrm{Med}(G^*)$ for any plane graph G. We call $(\mathrm{Med}(G))^*$ the *radial* graph of G and denote it by $\mathrm{Rad}(G)$. The dual graph of a C4P graph is bipartite. Thus $\mathrm{Med}(G)$ is a C4P graph and the radial graph of G is bipartite.

Conversely, any C4P graph H is the medial of some connected plane graph. The faces of H can be colored with 2 colors. An associated graph H' is defined as follows. Place a vertex in every face of H of a same color. Join two new vertices representing faces f_1 and f_2 by an edge through every vertex of H belonging to $f_1 \cap f_2$. It is easy to see that $H = \mathrm{Med}(H')$.

Let G be a C4P graph and C be a Petri circuit. The vertices of $\mathrm{Med}(G)$ corresponding to edges of C form a central circuit of $\mathrm{Med}(G)$.

Theorem 1 *For any C4P graph G, there is a C4P graph G' such that $G = \mathrm{Med}(G')$. Furthermore, a CC-edge-partition of G gives rise to an double edge-covering of G' by Petri circuits.*

Note that Nrs. 12-1, 12-2 of Figure 1 (the cuboctahedron and its twist) have the same v-, p- and CC-vectors, but they have different vectors of lengths of Petri circuits: (8^6) and $(18, 30)$, respectively. Also, they are medials for two very different polyhedra: the octahedron (or the cube) and, respectively, self-dual 7-vertex polyhedron with $v = (v_3 = 4 = p_3, v_4 = 3 = p_4) = p$.

For a polyhedron P, $Med(P)$ is defined to be the convex hull of the mid-points of all edges of P. If P^* denotes the polyhedron dual to P, then the skeleton of $\mathrm{Med}(P)$ is the medial graph of the skeleton of P.

Theorem 2 *The $\mathrm{Med}(G)$, where G is the skeleton of any Platonic or semiregular polyhedron, has balanced C4P graph.*

Above fact is proved by direct computation, given in Table 1 below.

So, in view of Theorem 1, five Platonic solids, the cuboctahedron, the icosidodecahedron and their duals are only finite edge-transitive polyhedra having double edge-covering by *elementary* Petri circuits, i.e. their medials are pure. Permitting locally-finite infinite planar 3-connected graphs, [8] show that other such graphs are only three regular partitions of Euclidean plane, the Kagome partition (Archimedean partition (3.6.3.6)), its dual [3.6.3.6] and several infinities of partitions of the hyperbolic plane.

4. Operations on octahedrites

By an *octahedrite*, denoted oc_n, we mean any 4-valent polyhedron with n vertices that possesses only 3-faces and 4-faces. Such polyhedra are so named because the octahedron is the smallest member oc_6 of the family. Other small examples include $oc_8 = \mathrm{APrism}_4$, and oc_9 is the smallest convex 4-valent polyhedron with odd number of vertices. Note that the medial graph of an oc_n is an oc_{2n}.

Let p_i denotes the number of i-faces. It is well-known that the p-vector of any 4-valent polyhedron satisfies $p_3 = 8 + \sum_{i=5}(i-4)p_i$. The Euler's equation $n - 2n + (p_3 + p_4) = 2$ implies that $p_3 = 8$ and $n = p_4 + 6$. It is proved in [7], see also page 282 of [6] that an oc_n exists if and only if $n = 6$ or $n \geq 8$.

The family of octahedrites is unique case of k-valent, $k > 3$, polyhedra with only a- and b-faces, for which p_a is fixed for given (k, b). For 3-valent polyhedra, there are 3 such cases, all with $b = 6$ and $(a, p_a) = (5, 12), (4, 6), (3, 4)$; the first of them is well-known family of *fullerenes* (see last Section below).

[11] computed the number of octahedrites oc_n for all $n \leq 50$: 18972. It is 1, 0, 1, 1, 2, 1, 5, 2, 8, 5, 12, 8, 25, 13, 30, 23, 51, 33, 76, 51, 109, 78, 144, 106, 218 for $n = 6, ..., 30$ and 150, 274, 212, 382, 279, 499, 366, 650, 493, 815, 623, 1083, 800, 1305, 1020, 1653, 1261, 2045, 1554, 2505 for $n = 31, ..., 50$.

Let $f_1, f_2, \ldots, f_m$ be a sequence of distinct faces of an octahedrite such that every pair f_i and f_{i+1} are adjacent, where the indices are taken modulo m. We also require f_i be adjacent to f_{i-1} and f_{i+1} on opposite edges if f_i is a 4-face. Such a ring of faces is called *zonal.* An operation called an *m-cutting* is defined as follows. Suppose that $f_1, f_2, \ldots, f_m$ is a zonal ring. We add a new vertex v_i to the common boundary of f_i and f_{i+1}, then we connect all new vertices by a cycle $v_1, v_2, \ldots, v_m$. This operation produces an oc_{n+m} from an oc_n so that all central circuits are preserved and one new central m-cycle is added. For example, Nrs. 12-1, 12-2 and 14-2 are, respectively, 6-, 6- and 8-cutting of the octahedron oc_6. Here and below, the examples are from Figure 1.

When a zonal ring consists entirely of 4-faces, an m-cutting is called an *m-elongation.* We could also view this operation as inserting a zonal ring of m 4-faces along a central m-cycle. For example, the elongated bipyramid $BPyr_4^{m+1}$, which is an oc_{4m+6}, is a 4-elongation of the octahedron oc_6 iterated $m+1$ times. Also Nr. 22-1 is an 8-elongation of Nr. 14-2.

Suppose in a given octahedrite there is a 3-face f that is adjacent only to 3-faces. We insert a new vertex in each edge of f and connect them to form a 3-cycle. We can iterate this operation to each newly created 3-cycle. After m times of iteration, the final product is called the 3-*decoration* of the original octahedrite The set of such octahedrites is preserved under this iteration operation. It contains an oc_n for any $n \equiv 0 \pmod 3$, actually the 3-decorations of the octahedron, i.e. APrism_3^m.

Claim 3 *A 3-decoration of any octahedrite oc_n is APrism_3^k with $k = m + \frac{n}{3}$.*

In fact, if there is a 3-face in an oc_n adjacent only to 3-faces, then it is easy to see that *each* of those 3 faces is, either adjacent to 3-faces only (and so our oc_n is the graph of the octahedron), or adjacent to two 4-faces. In the second case delete the original interior 3-face and the proof proceeds by induction.

Suppose in a given octahedrite there is a 4-face f that is adjacent only to 3-faces. We insert a new vertex in each edge of f and connect them to form a 4-cycle. We can iterate this operation to each newly created 4-cycle. After m times of iteration, the final product is called the 4-*decoration* of the original octahedrite. The set of such octahedrites is preserved under this iteration operation. It contains an oc_n, $n \geq 8$, for any $n \equiv 0, 1, 3 \pmod 4$, actually the 4-decorations of oc_8 (i.e. of APrism_4^{m+1}), of oc_9, of oc_{11}, and of the cuboctahedron.

5. CC-partitions for octahedrites

Let $C_1, \ldots, C_r$ be all central circuits of a given pure octahedrite. Since all circuits do not have self-intersections, $2n = \sum_{1 \leq i \leq r} |C_i| = \sum_{1 \leq i,j \leq r} |C_i \cup C_j| = 2 \sum_{1 \leq i < j \leq r} |C_i \cup C_j|$. Thus n is a sum of the even numbers $|C_i \cup C_j|$ and we have the following.

Claim 4 *The number n of vertices of any pure octahedrite oc_n is even.*

An equilinked polyhedron with CC-vector (a^r) and intersection vector $(c_0; c^{r-1})$ possesses $ra/2$ vertices and $a = c_0 + c(r-1)$. Consequently, n is odd if and only if r is odd and $a \equiv 2 \pmod 4$, both r and c_0 are odd. For example, we will show $n = 45$, $r = c_0 = 3$, $a = 30$, and $c = 12$ for APrism_4^9.

A pure equilinked polyhedron is called *r-uniform* if it has CC-vector (a^r) and intersection vector $(0; 2^{r-1})$. So $a = 2(r-1)$ and the number of vertices is $r(r-1)$. Such an oc_n exists for $r = 3, 4, 5$, and 6: the octahedron, the cuboctahedron, and Nrs. 20-1, 30-1 of Table 2; cf. last two with Nrs. 3 and 7 on Figure 2. Nr. 30-1, the icosidodecahedron and its twist (regular-faced pentagonal orthobirotunda) have all three, the same CC-vector (10^6) and the intersection vector $(0; 2^5)$. The family of $APrism_{r-2}^r$ provides examples for r-uniform equilinked polyhedron for any r.

The rhombicuboctahedron (see Nr 24-1 in Table 2) is an example of pure equilinked octahedrite that is not r-uniform: its CC-vector consists of six 8-cycles so that each has intersection vector $(0; 2^4, 0)$.

The effect of an m-cutting on the CC-vector can be understood as follows. A new central m-cycle is added and all other central circuits remain unchanged except that the length of a central circuit is increased by one each time it intersects the new m-cycle. For example, both, 6-elongation of the cuboctahedron and its twist, the $\text{Med}(oc_9)$, have the CC-vector $(6^2, 8^3)$.

Call an octahedrite *irreducible* if it is not an m-elongation of another octahedrite; call it *strongly irreducible* if it is not an m-cutting of another octahedrite. Clearly, an irreducible octahedrite has at most 24 central circuits since each circuit is incident to at least one 3-face.

Claim 5[4] *Any irreducible pure octahedrite has at most six central circuits. The upper bound is attained for Nr. 30-1.*

Table 1 supplies more examples of 4-valent pure irreducible polyhedra with larger number of central circuits.

In the row for $\text{Med}(\text{Prism}_m)$ of Table 1, the symbol t denotes the greatest common divisor of 4 and m, $m \geq 1$, while $\text{Med}(\text{Prism}_1)$ has the CC-vector (4^2) and the intersection vector $(1; 2)$. In the last line, $m \geq 2$. Note that $\text{Med}(\text{Prism}_4) = \text{Med}(\text{APrism}_3)$ is the cuboctahedron and that $\text{Med}(\text{Prism}_3)$ is the oc_9.

The numbers of all knot oc_n with $6 \leq n \leq 21$ are 0, 0, 1, 1, 0, 0, 2, 1, 1, 5, 6, 1, 6, 6, 13, 7. There are six non-pure equilinked oc_n with $n \leq 21$: APrism_3^4, APrism_4^4 (Nrs. 15-1, 20-2), and four equilinked oc_n (Nrs. 14-3, 18-1, 18-2, 20-3) are partitioned into two central $\frac{n}{2}$-circuits, with 3, 4, 4, 4 self-intersections, respectively (but two central circuits of Nr. 20-3 are not equal). There are 10 other oc_n with $n \leq 21$ having only self-intersecting central circuits, including Nr. 17-2 and one with three different sizes of central circuits.

For octahedrites of highest possible symmetry, [4] gives the following.

Theorem 6 *Any octahedrite oc_n with symmetry O_h can be obtained from the octahedron or cuboctahedron by replacing each central circuit by the same number $a \geq 1$ of parallel ones (i.e. $a-1$ "rail-roads").*

So, clearly, O_h-octahedrites are pure and balanced. More exactly, they have $n =$

$6a^2, 12a^2$, CC-vectors $(4a^{3a}),(6a^{4a})$ and intersection vectors $(0;2^{2a},0^{a-1})$, $(0;2^{3a},0^{a-1})$, respectively.

Note that Med^i*(tetrahedron)*, defined as i times iterated operation of taking of the medial, starting from the regular tetrahedron (cf. Table 1) is $O_h - oc_n$ with $n = 3 \times 2^i$, $i \geq 1$. It corresponds, for odd i, to $a = 2^{\frac{i-1}{2}}$ of the case $n = 6a^2$ above, and for even i, to $a = 2^{\frac{i-2}{2}}$ of the case $n = 12a^2$ above. Med^i(icosahedron) is a 15×2^i-vertex I_h-polyhedron with p-vector $(p_3 = 20, p_4 = 15(2^i - 2), p_5 = 12)$, $i \geq 1$, and its CC-vector can be found similarly.

In fact, [4] characterizes also more general variety of all octahedrites with octahedral symmetry, i.e. O_h or O; they have $6(a^2 + b^b)$ vertices and cases $b = 0, a$ correspond to the symmetry O_h. The smallest chiral one is Nr. 30-1 ((a,b)=(2,1)); consider also, for example, $oc_{78}(O)$ ((a,b)=(3,2)) and oc_{156} ((a,b)=(5,1)). Corresponding CC-vectors are (10^6), (52^3), (78^4) and the intersection vector for each central circuit is $(0;2^5)$, $(8;18^2)$, $(9;20^3)$, respectively. The deletion of all but one central circuit in those $oc_{78}(O)$ and $oc_{156}(O)$ reduces them to oc_8 and oc_9, respectively.

6. Prismatic polyhedra

We consider the following families of polyhedra:

Prism^k_m: a $Prism_m$ elongated $(k-1)$ times (i.e., a column of k m-prisms having skeleton $C_m \times P_{k+1}$);

Pyr^k_m: an m-pyramid elongated $(k-1)$ times (the apex has degree m);

BPyr^k_m: an m-bipyramid elongated $(k-1)$ times (the apexes have degree m);

APrism^k_m is a "column of k m-antiprisms", defined by APrism$^{2q-1}_m$:=Med(Pyr^q_m) and APrism$^{2q}_m$:=Med(BPyr^q_m) = Med(Prism^q_m).

It is clear that $(\text{Prism}^k_m)^* = \text{BPyr}^k_m$, $(\text{Pyr}^k_m)^* = \text{Pyr}^k_m$; also, BPyr$^k_4$ is an oc_{4k+2} and APrism^k_m is an $oc_{m(k+1)}$ for $m = 3$ and 4. In fact, APrism^k_m has $m(k+1)$ vertices and $2m$ 3-faces, $m(k-1)$ 4-faces, two m-faces; APrism$^k_3$ is an $oc_{3(k+1)}$ of the symmetry (for $k \geq 2$) D_{3d} and APrism$^k_4$ is an $oc_{4(k+1)}$ of symmetry D_{4d} for odd k and of symmetry D_{4h} for even $k > 2$.

All APrism$^k_3$ and APrism$^k_4$ with $k \equiv 0,1,3 \pmod 4$ are irreducible. However, in an APrism$^k_4$ with $k \equiv 2 \pmod 4$, the deletion of any of four central cycles produces an APrism$^\ell_3$, where $\ell = (3k-2)/4$.

Theorem 7 *(i) Any $APrism^k_m$ is an equilinked polyhedron with the CC-vector $((2m(k+1)/t)^t)$ and the intersection vector $(m(k+2-t)/t^2;(2m(k+2)/t^2)^{t-1})$, where $t = \gcd(m, k+2)$.*

(ii) The CC-vector of BPyr^k_m *with even m is $(m^k,(2k+2)^{m/2})$ with the intersection vectors $(0;2^{m/2},0^{k-1}),(0;2^{k-1+m/2})$.*

The cases $k = 1$ and 2 in (i) were given, essentially, as Theorems 3, 4 in [5]. (In fact, considered in those Theorems were graphs with cyclically ordered vertices $v_1, \ldots, v_n$ and edges (v_i, v_j) for all i and $j = i+1, i+2$ (addition modulo n). Clearly, any such graph is the skeleton of $APrism_{n/2}$ if n is even. Otherwise, this 4-regular graph is not planar, yet when drawn in the plane with vertices on a circle and chords (v_i, v_{i+2}) as edges it has the same CC-partition as $APrism_n$.

7. Face-regular and embeddable octahedrites

An octahedrite is called *face-regular* if either every 3-face is adjacent to exactly t_3 3-faces or every 4-face is adjacent to exactly t_4 4-face, where both t_3 and t_4 are fixed numbers. See [2], [3] for more on face-regular polyhedra with two types of faces, including part of following theorem.

Theorem 8 *Except two infinite families satisfying $t_3 = 0$ or $t_3 = 1$, all face-regular octahedrites are listed in Table 2.*

A complete description of all oc_n with $t_3 = 0$ seems difficult. We can give a few examples: any medial of an oc_n is such an oc_{2n}, and APrism_4^m for any $m \geq 2$. In fact, all oc_n with $t_3 = 0$ and $n \leq 21$ are Nrs. 12-1, 16-1, 16-3, 20-1, 20-2 and two with $n = 18$.

There are also infinitely many octahedrites with $t_3 = 1$. For example, those oc_n with $n = 8s + 14$ and $6s + 16$. They are derived as 8- or 6-elongations of Nrs. 14-2 and 16-2 iterated s times. All oc_n with $t_3 = 1$ and $n \leq 21$ are Nrs. 14-2, 14-3, 16-2, 16-4, 18-1, one more with $n = 18$ and one with $n = 20$.

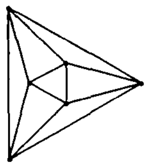
The case of $t_3 = 2$ is much simpler.

Claim 9 *The only polyhedra oc_n with $t_3 = 2$ are either* APrism_4 *or the family $BPyr_4^m$, $m \geq 2$.*

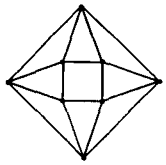
Proof Let T_0 be a 3-face in an oc_n with $t_3 = 2$. Then T_0 is adjacent to two 3-faces T_1 and T_2. These 3-faces are adjacent to other 3-faces. There are two cases: T_1 and T_2 do or do not have a common adjacent 3-face. In the first case, we obtain a configuration of four 3-faces surrounded by four 4-faces. This configuration generates the family BPyr_4^m. In the second case, we can only obtain APrism_4. □

A well-known metric space on the vertices of a polyhedron P can be defined in terms of the shortest-path metric on its skeleton. Then P is said to be *embeddable* if this metric space can be embedded isometrically into a hypercube H_m or into a half-hypercube $\frac{1}{2}H_m$. See [3] and references there for more on embeddability of polyhedra. All octahedrites or their duals, that are known to be embeddable, are indicated in the last two columns of Table 2.

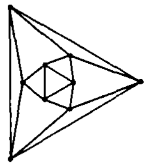
Figure 1 includes all oc_n with $n \leq 13$ and selected ones for larger n, such as all Gaussian ones with $n \leq 15$, all pure ones with $n \leq 17$, all equilinked (but non-pure and non-Gaussian) ones with $n \leq 21$, and all those listed in Table 2. Note that the symmetry group of the first nine octahedrites (Nrs. 6 to 30-1) of Table 2 (see also Figure 1) has exactly two orbits on faces, for all but Nrs. 14-3 and 22-1. For each item P, Figure 1 indicates the number of vertices, CC-vector and the order of the symmetry group $\mathrm{Sym}(P)$.



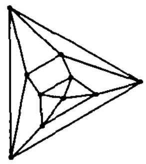
Nr.6 (4^3)
Groupsize: 48



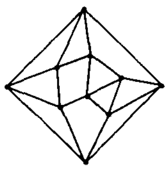
Nr.8 (16)
Groupsize: 16



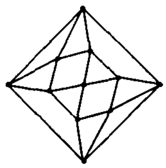
Nr.9 (18)
Groupsize: 12



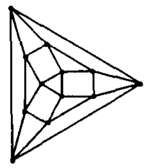
Nr.10-1 $(4^2, 6^2)$
Groupsize: 16



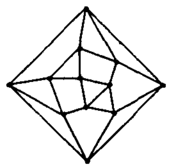
Nr.10-2 (6, 14)
Groupsize: 4



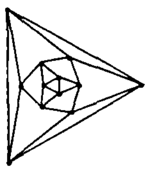
Nr.11 $(6^2, 10)$
Groupsize: 4



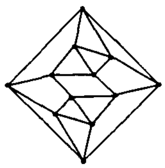
Nr.12-1 (6^4)
Groupsize: 48



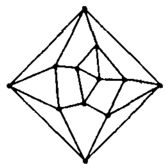
Nr.12-2 (6^4)
Groupsize: 12



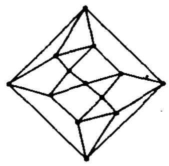
Nr.12-3 (24)
Groupsize: 12



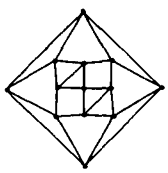
Nr.12-4 (24)
Groupsize: 4



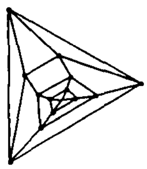
Nr.12-5 (6, 18)
Groupsize: 2



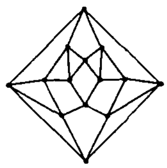
Nr.13-1 (26)
Groupsize: 2



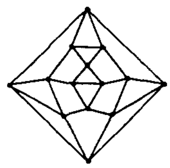
Nr.13-2 $(6^2, 14)$
Groupsize: 4



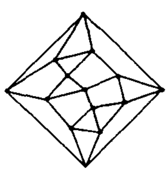
Nr.14-1 $(4^3, 8^2)$
Groupsize: 16



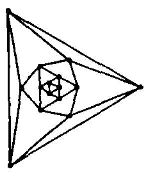
Nr.14-2 $(6^2, 8^2)$
Groupsize: 16



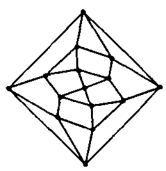
Nr.14-3 (14, 14)
Groupsize: 8



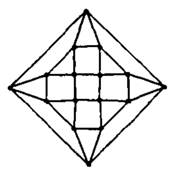
Nr.14-4 (28)
Groupsize: 2



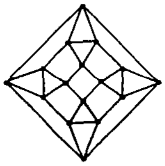
Nr.15-1 (10^3)
Groupsize: 12



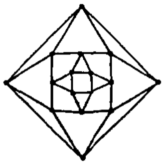
Nr.15-2 (30)
Groupsize: 2



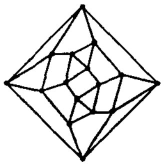
Nr.16-1 (8, 24)
Groupsize: 16



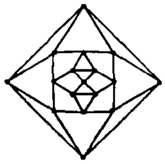
Nr.16-2 (6^4, 8)
Groupsize: 16



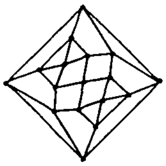
Nr.16-3 (32)
Groupsize: 16



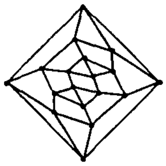
Nr.16-4 (6^2, 20)
Groupsize: 4



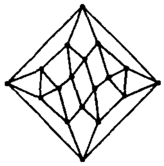
Nr.17-1 (34)
Groupsize: 4



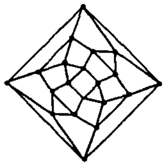
Nr.17-2 (10, 24)
Groupsize: 4



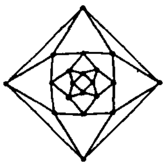
Nr.18-1 (18, 18)
Groupsize: 8



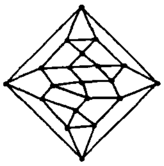
Nr.18-2 (18, 18)
Groupsize: 2



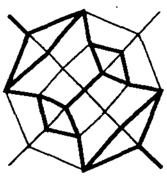
Nr.20-1 (8^5)
Groupsize: 8



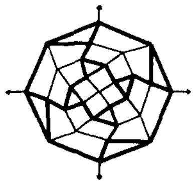
Nr.20-2 (20, 20)
Groupsize: 16



Nr.20-3 (20, 20)
Groupsize: 1

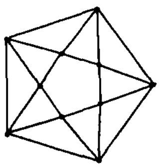


Nr.22-1 (8^4, 12)
Groupsize: 8

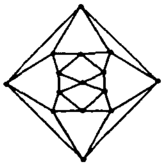


Nr.30-1 (10^6)
Groupsize: 24

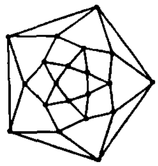
Figure 1: Small octabedrites



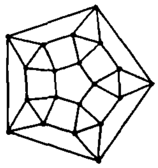
Nr.1 (20)
Group: D_{5d}
(t_3, t_5)=(2, 0)



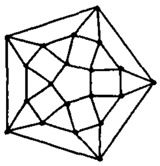
Nr.2 (30)
Group: D_{5h}
(t_3, t_4, t_5)=(0, 0, 0)



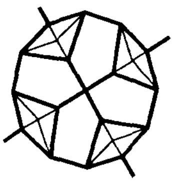
Nr.3 (8^5)
Group: D_{5d}
(t_3, t_4, t_5)=(0, 2, 0)



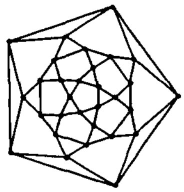
Nr.4 (6^5, 10)
Group: D_{5h}
(t_3, t_4, t_5)=(0, 1, 0)



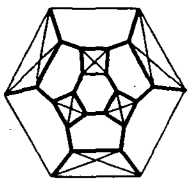
Nr.5 (10, 30)
Group: D_{5d}
(t_3, t_4, t_5)=(0, 0, 0)



Nr.6 (4^4, 8^2, 12)
Group: D_{4h}
(t_3, t_5)=(2, 2)



Nr.7 (10^6)
Group: I_h
(t_3, t_5)=(0, 0)



Nr.8 (4^6, 12^3)
Group: O_h
(t_3, t_6)=(2, 3)

Figure 2: Some other face-regular 4-valent polyhedra

8. Fullerenes with pure medials

A graph G is called *Eulerian* if there is a closed walk traversing every edge of G exactly once. It is a well-known fact that a graph is Eulerian if and only if it is connected and every vertex has even degree. Let G be a plane drawing of an Eulerian graph. We do not require G to be planar, so edges may have extra crossings. To avoid trivialities, we assume that the degree of every vertex is at least 4. Then all edges adjacent to a vertex x can be enumerated in the clockwise direction as $e_1, e_2, \ldots, e_k$ where k is the degree of x in G. (A self-loop is enumerated twice.) For any edge e_i, $1 \le i \le k$, the edges e_{i+1}, e_{i-1}, and $e_{(2i+k)/2}$ are called, respectively, the *left*, the *right*, and the *opposite* neighbor of e_i (addition modulo k).

A circuit C of G is called a *Petri circuit* if in tracing the circuit we alternately select as next edge the left neighbor and the right neighbor. A circuit C of G is called a *central circuit* if every edge of C is the opposite neighbor of its preceding edge.

The concept of a CC-partition can be generalized naturally to plane drawings of Eulerian graphs. For example, a certain drawing of the 4-dimensional hypercube $K_2^4 = C_4 \times C_4$ has a CC-partition into two circuits, each is of length 16 and is Hamiltonian.

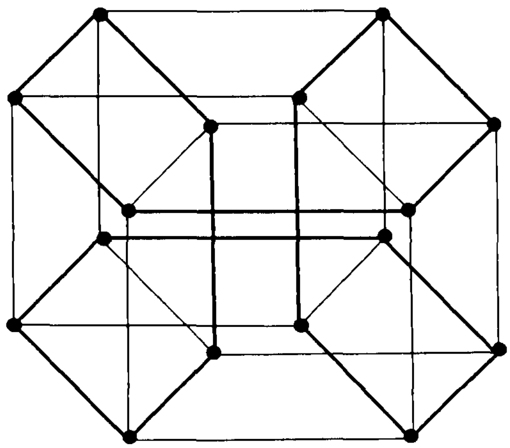


Figure 3: A CC-partition for the 4-dimensional hypercube

Actually, Petri circuits can be defined for any drawing of a graph G in the plane with extra crossings permitted. Then all Petri circuits of G form a covering of the edge set of G.

We call this covering the *Petri* covering. Since once a Petri circuit begins with an edge in one of the two directions it extends in a unique way. Hence each edge is covered twice by the Petri covering (see Theorem 1 above).

A simple polyhedron having only 5-faces and 6-faces is called a *fullerene*. Brinkmann in [1] computed all fullerenes P with at most 200 vertices, such their Petri circuits (and so, central circuits of Med(P)) are cycles. Table 3 is obtained by exploring his data. In column 5 the intersection vectors of Petri circuits are exhibited only if there are at most two different ones amongst them. In the first three cases not exhibited, i.e., Nrs. $60_3, 72_1$

and 80_2, there are three different intersection vectors: $(0;4,2^8)$, $(0;2^9)$, $(0;2^8,0)$; $(0;4,2^9)$, $(0;2^{10})$, $(0;2^9,0)$; $(0;2^{11})$, $(0;2^{10},0)$, $(0;2^9,2^2)$, respectively.

In column 3, the numbers t_5 (and/or t_6) appear if any 5-face (respectively, 6-face) of fullerene P is adjacent to exactly t_5 5-faces (respectively, t_6 6-faces), i.e. if P is face-regular (see previous Section). Last two columns are about embedding of P and/or P^* (cf. Table 2). All four known fullerenes P with embeddable P^* appear there.

Claim 3 implies that any n-vertex fullerene P with Med(P) being pure, has $n \equiv 0 \pmod 4$, since Med(P) has $\frac{3n}{2}$ vertices and their number should be even. On the other hand, such fullerene exists for any $n \leq 220, n \equiv 0 \pmod 4$, except n=24,32,40,52,64,96, and supposed to exist for any $n \geq 100, n \equiv 0 \pmod 4$.

The fullerenes with pure medials are not necessarily of high symmetry. For example, T_d-fullerenes with 40 and 76 vertices are not in this list. Fullerenes $P = 84_3$ and 100_2 have Sym(P) of order 2 and, respectively, 42 and 57 orbit of vertices. Starting with 188 vertices, appear fullerenes with trivial symmetry. But any fullerene with the highest symmetry I_h has pure medial. It is well-known that all fullerenes P with full icosahedral symmetry I_h have $20a^2$ or $60a^2$ vertices, where a is any natural number. It was shown in [4] that Med(P) has, for any I_h-fullerene, CC-vectors $(10a^{6a})$, $(18a^{10a})$ and intersection vectors $(0;2^{5a},0^{a-1})$, $(0;2^{9a},0^{a-1})$, respectively.

Note that [9] consider another way to get 4-regular graphs (and so, alternating links) from fullerenes: fix an edge-coloring and consider colored edges as digons.

Remind that a polyhedron is said to be *simple*, when every vertex has degree 3, and its *p-vector* $(..., p_i, ...)$ enumerate the numbers p_i of its faces having i sides. On page 296 of [6] Grünbaum states, "No example is known to disprove the conjecture that the numbers p_i, together with the specification of the different types of closed Petri-curves and their numbers, determine the combinatorial type of *simple* 3-polytopes". The pairs $(56_1, 56_2)$, $(60_1, 60_2)$, $(88_1, 88_2)$, $(88_3, 88_4)$ and the triple $(84_1, 84_2, 84_3)$ of fullerenes in Table 3 provide examples to disprove above conjecture.

Table 1: CC-vectors of medial polyhedra of Platonic and semiregular polyhedra

n	Polyhedron	CC-vector	Int. vectors
6	*octahedron=Med(tetrahedron)*	(4^3)	$(0;2^2)$
12	*cuboct.=Med(oct.)=Med(cube)*	(6^4)	$(0;2^3)$
30	*icosido.=Med(ico.)=Med(dode.)*	(10^6)	$(0;2^5)$
24	*rhombicub.=Med(cuboctahedron)*	(8^6)	$(0;2^4,0)$
60	*rhombicosa.=Med(icosidode.)*	(10^{12})	$(0;2^5,0^6)$
48	*Med(rhombicuboctahedron)*	(12^8)	$(0;2^6,0)$
120	*Med(rhombicosidodecahedron)*	(20^{12})	$(0;2^{10},0)$
72	*Med(trunc. cuboctahedron)*	(18^8)	$(0;6,2^6)$
180	*Med(trunc. icosidodecahedron)*	(30^{12})	$(0;10,2^{10})$
18	*Med(trunc. tetrahedron)*	(12^3)	$(0;6^2)$
36	*Med(trunc. octahedron)*	(12^6)	$(0;4,2^4)$
36	*Med(truncated cube)*	(18^4)	$(0;6^3)$
90	*Med(trunc. icosahedron)*	(18^{10})	$(0;2^9)$

90	*Med(trunc. dodecahedron)*	(30^6)	$(0;6^5)$
60	*Med(snub cube)*	(30^4)	$(3;8^3)$
150	*Med(snub dodecahedron)*	(50^6)	$(5;8^5)$
3m	$Med(Prism_m) = APrism_m^2$	$(6m/t)^t$	$(m(4-t)/t^2;(8m/t^2)^{t-1})$
4m	$Med(APrism_m), m \neq 3$	$(2m;6m)$	$(0;2m),(2m;2m)$

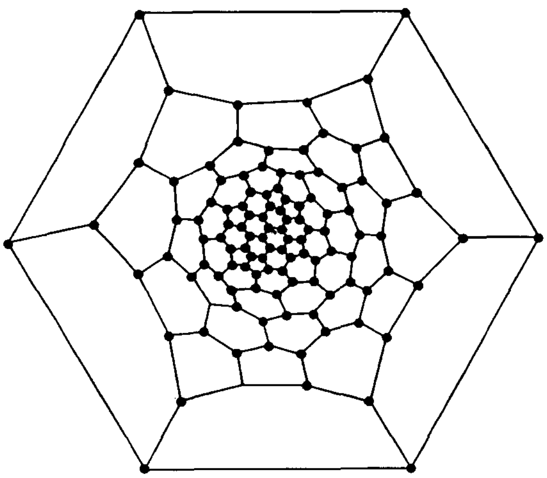
Table 2: The face-regular (but for $(t_3,t_4) = (1,-),(0,-)$ only minimal ones) and known embeddable n-vertex octahedrites $P = n - i$

P	Sym(P)	t_3,t_4	Polyhedron P	CC(P)	Emb. P	Emb. P^*
6	O_h	3	$BPyr_4 = APrism_3$	(4^3)	$1/2H_4$	H_3
8	D_{4d}	2,0	$APrism_4$	(16)	$1/2H_5$	–
10-1	D_{4h}	2,2	$BPyr_4^2$	$(4^2,6^2)$	$1/2H_6$	H_4
12-1	O_h	0,0	cuboct.$=APrism_4^2$	(6^4)	–	H_4
14-1	D_{4h}	2,3	$BPyr_4^3$	$(4^3,8^2)$	$1/2H_8$	H_5
14-2	D_{4h}	1,2	cut. Nr.6	$(6^2,8^2)$	–	–
14-3	D_{2d}	1,2	decor. (cuboct.)*	(14, 14)	–	–
22-1	D_{2d}	1,3	elong. Nr.14-2	$(8^4,12)$	–	–
30-1	O	0,3	cut. Nr.20-1	(10^6)	–	–
9	D_{3d}	–,0	$APrism_3^2$	(18)	$1/2H_6$	–
10-2	D_2	–,1		$(6;14)$	$1/2H_6$	–
12-2	D_{3h}	–,1	tw. cuboctahedron	(6^4)	–	–
12-3	D_{3d}	–,2	$APrism_3^3$	(24)	–	–
12-4	D_2	–,2		(24)	–	–
4m+2	D_{4h}	2,–	$BPyr_4^m, m \geq 4$	$(4^m,(2m+2)^2)$	$1/2H_{2m+2}$	H_{m+2}
16-2	D_{4h}	1,–	el. 10-1=tw. 16-1	$(6^4,8)$	$1/2H_8$	–
16-4	D_2	1,–	elong. Nr. 10-2	$(6^2;20)$	$1/2H_8$	–
16-1	D_{4d}	0,–	Med(Nr.8)=cut.Nr.8	$(8;24)$	–	–
16-3	D_{4d}	0,–	$APrism_4^3$	(32)	–	–
20-1	D_{2d}	0,–	cut. Nr.12-1	(8^5)	–	–
24-1	O_h	0,–	rhombicu.=el. 16-2	(8^6)	$1/2H_{10}$	–

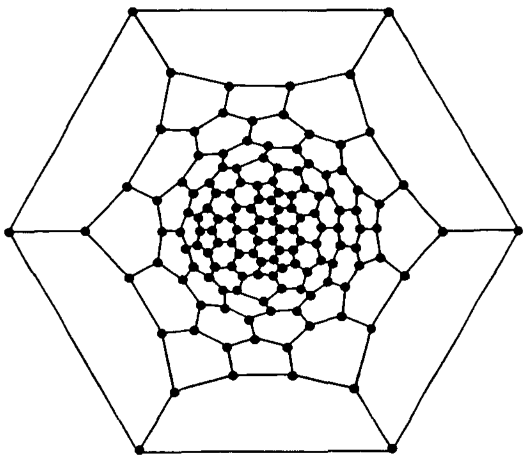
Table 3: All n-vertex fullerenes $P = n_i$ with $n < 116$ and pure Med(P)

P	Sym(P)	t_5,t_6	CC(med(P))	Int.vectors	Emb. P	Emb. P^*
20_1	I_h	5	(10^6)	$(0;2^5)$	$1/2H_{10}$	$1/2H_6$
28_1	T_d	3,0	(12^7)	$(0;2^6)$	–	$1/2H_7$
36_1	D_{6h}	2,–	$(14^6,12^2)$	$(0;2^7),(0;2^6,0)$	–	$1/2H_8$
48_1	D_3	–,3	(16^9)	$(0;2^8)$	–	–
56_1	T_d	2,4	$(18^4,16^6)$	$(0;2^9),(0;2^8,0)$	–	–
56_2	D_{3d}	1,–	$(18^4,16^6)$	$(0;2^9),(0;2^8,0)$	–	–
60_1	I_h	0,3	(18^{10})	$(0;2^9)$	–	$1/2H_{10}$
60_2	D_3	–,–	(18^{10})	$(0;2^9)$	–	–
60_3	D_{2h}	–,–	$(20^2,18^6,16^2)$		–	–
68_1	T_d	1,4	$(20^3,18^8)$	$(0;2^{10}),(0;2^9,0)$	–	–

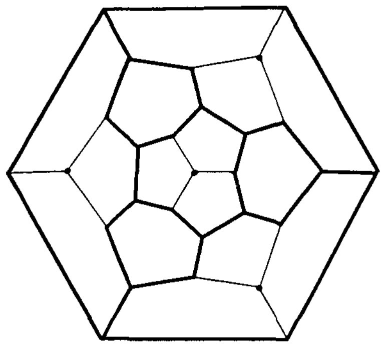
72_1	C_{2v}	–,–	$(22^2, 20^5, 18^4)$		–	–
76_1	D_{2d}	1,–	$(22^4, 20^7)$	$(0;4,2^9),(0;2^{10})$	–	–
80_1	I_h	0,4	(20^{12})	$(0;2^{10},0)$	$1/2H_{22}$	–
80_2	D_{3d}	1,–	$(22^3, 20^6, 18^3)$		–	–
84_1	D_{6h}	0,–	$(22^6, 20^6)$	$(0;2^{11}),(0;2^{10},0)$	–	–
84_2	D_{3d}	0,–	$(22^6, 20^6)$	$(0;2^{11}),(0;2^{10},0)$	–	–
84_3	C_2	–,–	$(22^6, 20^6)$	$(0;2^{11}),(0;2^{10},0)$	–	–
84_4	D_{2h}	–,–	$(24^2, 22^2, 20^8)$		–	–
84_5	C_{2h}	–,–	$(24^2, 22^5, 20^2, 18^3)$		–	–
88_1	T	0,–	(22^{12})	$(0;2^{11})$	–	–
88_2	D_{2h}	–,–	(22^{12})	$(0;2^{11})$	–	–
88_3	C_{2v}	–,–	$(24^4, 22^4, 20^4)$		–	–
88_4	C_{2v}	–,–	$(24^4, 22^4, 20^4)$		–	–
92_1	T_d	2,–	$(24^4, 20^9)$	$(0;2^{12}),(0;2^{10},0^2)$	–	–
92_2	T_h	1,–	$(24^6, 22^6)$	$(0;4,2^{10}),(0;2^{11})$	–	–
100_1	C_3	–,–	$(24^7, 22^6)$	$(0;2^{12}),(0;2^{11},0)$	–	–
100_2	C_s	–,–	$(26^2, 24^6, 22^2, 20^3)$		–	–
104_1	C_{2v}	–,–	$(26^4, 24^5, 22^4)$		–	–
108_1	D_{3d}	1,–	$(26^3, 24^2, 22^9)$		–	–
108_2	D_{2d}	1,–	$(28, 26^4, 24^8)$		–	–
112_1	T_d	0,–	(24^{14})	$(0;2^{12},0)$	–	–
112_2	C_{2h}	0,–	$(26^3, 24^8, 22^3)$		–	–
112_3	D_{2h}	–,–	$(28^2, 26^2, 24^4, 22^6)$		–	–
112_4	D_{2h}	–,–	$(28^2, 26^4, 24^4, 20^4)$		–	–
140_1	I	0,5	(28^{15})	$(0;2^{14})$	–	–
180_1	I_h	0,–	(30^{18})	$(0;2^{15},0^2)$	–	–



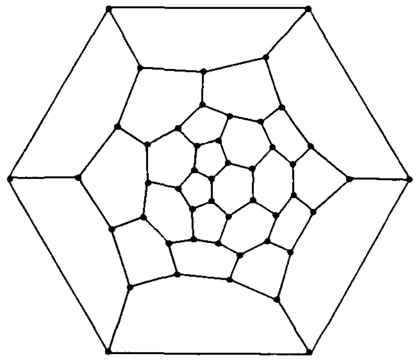
Nr.112_4 $(28^2, 26^4, 24^4, 20^4)$
Group: D_{2h}



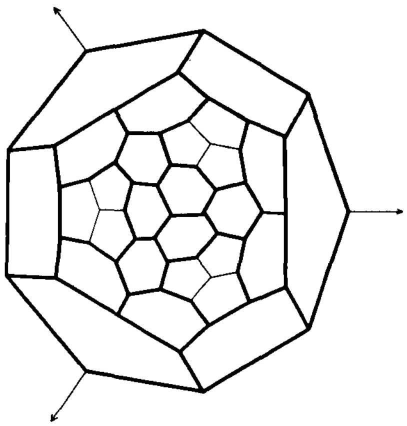
Nr.140_2 $(30^3, 26^9, 24^4)$
Group: D_{3d}



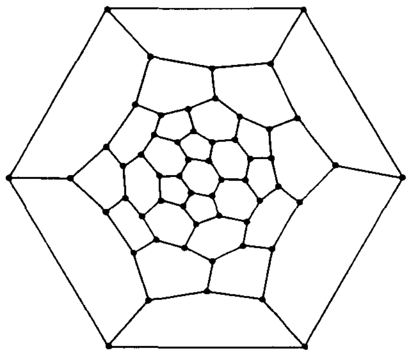
Nr.28_1 (12^7)
Group: T_d



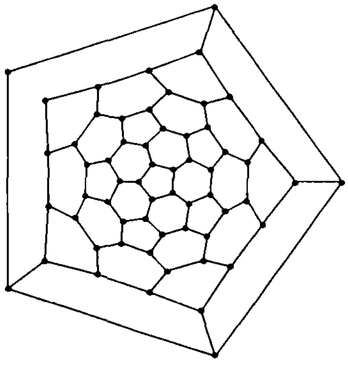
Nr.48_1 (16^9)
Group: D_3



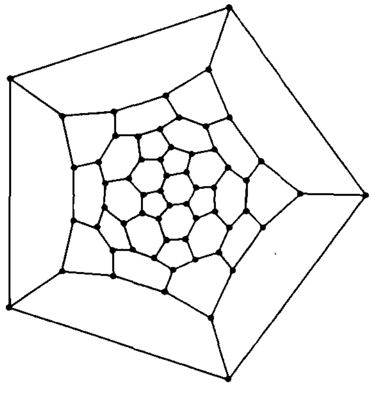
Nr.56_1 $(18^4, 16^6)$
Group: T_d



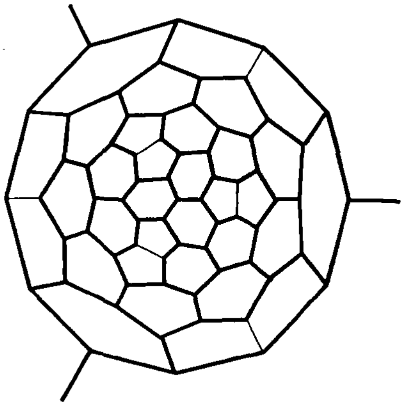
Nr.56_2 $(18^4, 16^6)$
Group: D_{3d}



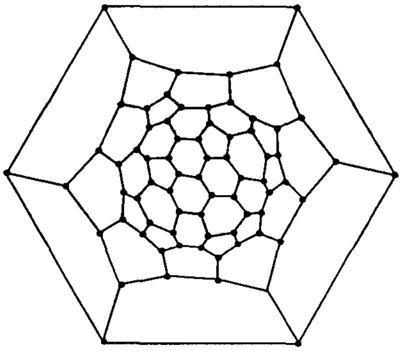
Nr.60_1 (18^{10})
Group: I_h



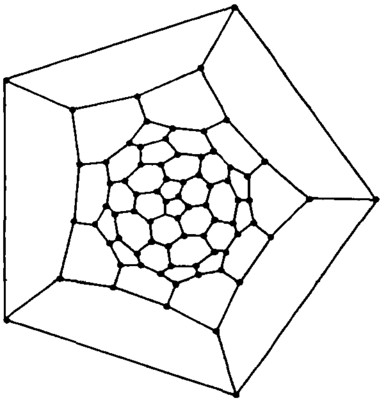
Nr.60_2 (18^{10})
Group: D_3



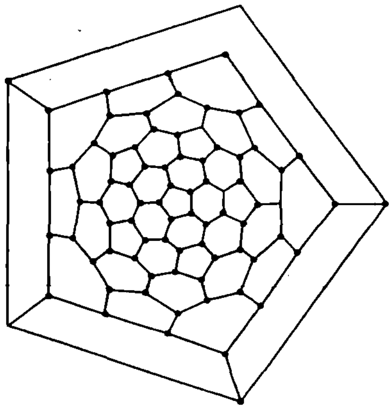
Nr.68$_1$ (20^3, 18^8)
Group: T_d



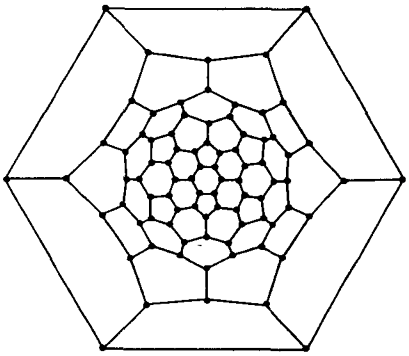
Nr.72$_1$ (22^2, 20^5, 18^4)
Group: C_{2v}



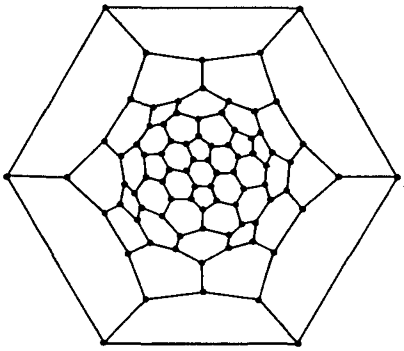
Nr.76$_1$ (22^4, 20^7)
Group: D_{2d}



Nr.80$_1$ (20^{12})
Group: I_h



Nr.84$_1$ (22^6, 20^6)
Group: D_{6h}



Nr.84$_1$ (24^2, 22^2, 20^8)
Group: D_{2h}

Figure 4: Some fullerenes from Table 3

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类八面体与中点多面体的中央回路覆盖

Michel Deza[1]，黄 大 原[2]，李 国 伟[3]

(1. 法国巴黎高等师范学院； 2. 台湾交通大学应用数学系； 3. 台湾中央研究院数学研究所)

摘 要: 类八面体是顶点度数为 4, 而仅具有 3 边面与 4 边面的多面体. 本论文研究把类八面体、中点多面体及其它相关多面体，分解为无共享边的中央回路. 除新结果外，也将此方面的已知结果整理为表与图，方便想进一步研究者参考所需.