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Conditional Stability of Stochastic Volterra Equations with Anticipating Kernel *

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Abstract: This paper introduces some concepts of conditional stability of stochastic Volterra equations with anticipating kernel. Sufficient conditions of these types of stability are established via Lyapunov function.

Key words: stochastic Volterra equations; conditional stability; anticipating stochastic calculus; Skorohod integral.

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1. Introduction

The theory of stochastic integration of processes which is not necessarily adapted has been developed by many authors. It makes possible to investigate stochastic differential equations with anticipating coefficients. In particular, anticipating stochastic Volterra differential equations have been discussed recently in [12] and [13]. Which have very important application in finance theory, see [5] and [6]. Clearly the problem is on the stochastic integrals which are not adapted, and therefore one cannot use as usual the Itô integral to interpret the equation. Hence the Skorohod integrals (cf.[14]) be used. On the other hand, stability analysis of various kinds of non-anticipating stochastic differential equations have been well studied, which is very important both in theory and in applications of stochastic dynamic systems, see [1],[7],[8],[15],[16],[17] and [18]. However, as we know, the stability problem has not been discussed for the anticipating stochastic equations. The aim of this paper is to give a very beginning discussion of stability analysis to stochastic Volterra equations with anticipating kernel:

$$X_t = x_0 + \int_{t_0}^{t} F(X_s)\mathrm{d}s + \sum_{i=1}^{k}\int_{t_0}^{t} G_i(H_t, X_s)\delta W_s^i, \tag{1}$$

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where $x_0 \in L^q(\Omega, \mathcal{F}_0, P; R^d)$; $F \in C^1(R^d, R^d)$ and $G_i \in C^{1,0}(R^p \times R^d, R^d), i = 1, 2, \cdots, k$, H_t is a given p-dimensional progressively measurable process, and the last stochastic integral is in the Skorohod sense. The concept of conditional stability, conditional uniform asymptotically stability and weak conditional exponential stability are introduced. Some sufficient conditions of these types of stability are given.

1. Preliminaries

Let $\Omega = C(R_+; R^k)$ be equipped with the topology of uniform convergence on compact subsets of R_+, $\mathcal{F}$ be the Borel field over Ω, and P denote the standard Wiener measure on $(\Omega, \mathcal{F})$, i.e., $\{W_t(\omega) = \omega(t), t \geq 0\}$ is a standard $(EW_tW_t' = tI)$ Wiener process under P. $\mathcal{S}$ denote the subset of $L^2(\Omega)$ consisting of those random variables F which take the form

$$F = f(W(h_1), \cdots, W(h_n)),$$

where $n \in N; h_1, \cdots, h_n \in H = L^2(R_+; R^k); f \in C_b^\infty(R^n)$. Here

$$W(h) = \int_0^\infty \langle h(t), \mathrm{d}W_t \rangle.$$

If $F \in \mathcal{S}$, then its gradient is defined by

$$D_t^j F = \sum_{i=1}^n \frac{\partial f}{\partial x_i}(W(h_1), \cdots, W(h_n)) h_i^j(t)$$

DF will stand for the k-dimensional process $\{D_tF = (D_t^1 F, \cdots, D_t^k)'; t \geq 0\}$.

It is known (proposition 2.1 in [13]) that $D^j, j = 1, \cdots, k(resp.D)$ is a closable unbounded operator from a subset of $L^2(\Omega)$ into $L^2(R_+ \times \Omega)$ (resp. $L^2(R_+ \times \Omega; R^k)$). We identify D^j (resp.D) with its closed extension, and denote its domain by $\mathcal{D}_j^{1,2}$ (resp.$\mathcal{D}^{1,2}$). It is easy to see that $\mathcal{D}^{1,2} = \cap_{j=1}^k \mathcal{D}_j^{1,2}$. $\mathcal{D}_j^{1,2}$ and $\mathcal{D}^{1,2}$ are the closures of $\mathcal{S}$ with respect to, respectively the norms

$$||F||_{j,1,2} = ||F||_2 + ||||D^j F||_{L^2(R_+)}||_2$$

and

$$||F||_{1,2} = ||F||_2 + \sum_{i=1}^k ||||D^i F||_{L^2(R_+)}||_2.$$

More generally, we can define $\mathcal{D}_j^{1,p}$ and $\mathcal{D}^{1,p}$ similarly, the closures of $\mathcal{S}$ with respect to the norms

$$||F||_{j,1,p} = ||F||_p + ||||D^j F||_{L^2(R_+)}||_p,$$

and

$$||F||_{1,p} = ||F||_p + \sum_{i=1}^k ||||D^i F||_{L^2(R_+)}||_p,$$

respectively. And furthermore let $\mathcal{D}_j^{2,p}$ and $\mathcal{D}^{2,p}$ be the closures of $\mathcal{S}$ with respect, to respectively the norms

$$||F||_{j,2,p} = ||F||_p + ||||D^jD^jF||_{L^2(R_+^2)}||_p$$

and

$$||F||_{2,p} = ||F||_p + ||\sum_{i,j=1}^{k} ||D^iD^jF||_{L^2(R_+^2)}||_p.$$

Define

$$\mathcal{L}_j^{l,p} = L^p(R_+, \mathrm{d}t; \mathcal{D}_j^{l,p}), j = 1, \cdots, k; l = 1 \ or\ 2$$

and

$$\mathcal{L}^{l,p} = L^p(R_+, \mathrm{d}t; \mathcal{D}^{l,p}), l = 1 \ or\ 2.$$

The Skorohod integral $(D^{j,*}, \mathrm{Dom}D^{j,*})$ is defined as the adjoint of D^j, i.e., it is the closed unbounded linear operator from $L^2(R_+ \times \Omega)$ into $L^2(\Omega)$ which is defined as follows:

(i) $\mathrm{Dom}D^{j,*}$ is the set of those $u \in L^2(R_+ \times \Omega)$ to which we can associate a constant c such that :

$$|E\int_0^\infty D_t^jFu_t\mathrm{d}t| \le c||F||_2, \forall F \in \mathcal{S}.$$

(ii) If $u \in DomD^{j,*}$, $D^{j,*}(u)$ is the unique element of $L^2(\Omega)$ which is such that

$$E[FD^{j,*}(u)] = E[\int_0^\infty D_t^jFu_t\mathrm{d}t], \forall F \in \mathcal{S}.$$

We know that $\mathcal{L}_j^{1,2} \subset \mathrm{Dom}D^{j,*}$, and if $u \in \mathcal{L}_j^{1,2}$ then $u1_{[0,t]} \in \mathcal{L}_j^{1,2}$, hence we can restrict the operator $D^{j,*}$ to $\mathcal{L}_j^{1,2}$, and define the Skorohod integral by

$$\int_{t_0}^t u_s\delta W_s^j = D^{j,*}(u1_{[t_0,t]}).$$

We recall the anticipative Itô formula by Pardoux and the existence theorem which will be needed. The proofs can be found in [12] and [13].

Theorem 2.1 *Let $V \in C_b^2(R^d))$, x_0 be a d-dimensional random vector, $A_t, B_t^1, \cdots, B_t^k$ be d-dimensional random processes such that*

(1) $x_0^j \in \mathcal{D}^{1,4}, j = 1, \cdots, d;$

(2) $A^j \in \mathcal{D}^{1,4}, j = 1, \cdots, d;$

(3) $B_l^j \in \mathcal{L}^{2,p}, l = 1, \cdots, k$ and $j = 1, \cdots, d$, for some $p > 4$.

Let $X_t = x_0 + \int_{t_0}^t A_s\mathrm{d}s + \int_{t_0}^t B_s^j\delta W_s^j; t \ge 0$. Then

$$V(X_t) = V(x_0) + \int_{t_0}^t (V'(X_s), A_s)\mathrm{d}s + \sum_{j=1}^k \int_{t_0}^t (V'(X_s), B_s^j)\delta W_s^j + \frac{1}{2}\sum_{j=1}^k \int_{t_0}^t (V''(X_s)(\nabla^jX)_s, B_s^j)\mathrm{d}s, \tag{2}$$

where

$$(\nabla^jX)_t = 2D_t^jx_0 + 2\int_{t_0}^t D_t^jA_s\mathrm{d}s + 2\sum_{i=1}^k \int_{t_0}^t D_t^jB_s^i\delta W_s^j + B_t^j,$$

where

$$D_t^j x_0 = (D_t^j x_0^1, \cdots, D_t^j x_0^d), D_t^j A_s = (D_t^j A_s^1, \cdots, D_t^j A_s^d),$$
$$D_t^j B_s^l = (D_t^j B_s^{l,1}, \cdots, D_t^j B_s^{l,d}).$$

Theorem 2.2[12] *Suppose there exists $q > p$, B a bounded subset of R^p and $K > 0$ such that: $x_0 \in L^q(\Omega, \mathcal{F}_0, P; R^d)$, $H_t \in B, a.s., \forall t \geq 0$, $H \in (\mathcal{L}^{1,2}); |D_s H_t| \leq K a.s., 0 \leq s \leq t$, and increasing and Lipschitz conditions as follows*

$$|F(x)| + \sum_{i=1}^{k} |G_i(h,x)| + \sum_{i=1}^{k} |\frac{\partial G_i}{\partial h}(h,x)| \leq K(1+|x|)$$

and

$$|F(x) - F(y)| + \sum_{i=1}^{k} |G_i(h,x) - G_i(h,y)| + \sum_{i=1}^{k} |\frac{\partial G_i}{\partial h}(h,x) - \frac{\partial G_i}{\partial h}(h,y)| \leq K|x-y|$$

for $0 \leq s \leq t, h \in B, x, y \in R^d$. Then the stochastic equations (1.1) has a unique solution X_t in $\cap_{t>0} L^q_{prog}(\Omega \times (t_0,t))$, where $L^q_{prog}(\Omega \times (t_0,t))$ stands for the space $L^q(\Omega \times (t_0,t), \mathcal{P}_t, P \times \lambda)$, here $\mathcal{P}_t$ denotes the σ-field of progressively measurable subsets of $\Omega \times (t_0,t)$ and λ denotes the Lebesgue measure on (t_0,t).

2. Stability

Definition 3.1 *Let X_t denote any solution of (1.1). The equations (1.1) is said to be:*

(i) conditionally stable, if for any $\varepsilon > 0, t_0 \in R_+$, there exists a $\delta(t_0,\varepsilon) > 0$ such that

$$E[|X_t| \mid \mathcal{F}_{]t_0,t]^c}] \leq \varepsilon, \text{a.s.}, \quad \forall t > t_0,$$

whenever $E[|x_0| \mid \mathcal{F}_{]t_0,t]^c}] \leq \delta(t_0,\varepsilon), \forall t > 0$, here $\mathcal{F}_{]t_0,t]^c}$ denotes the σ-field generated by the increments of a standard Wiener process on $R_+ -]t_0,t]$;

(ii) conditional uniformly stable, if the δ above is independent of t_0;

(iii) conditional quasi-equi-asymptotically stable if, for any $\varepsilon > 0, t_0 \in R_+$, there exist $\delta(t_0) > 0$ and $T(t_0,\varepsilon) > 0$ such that if $E[|x_0| \mid \mathcal{F}_{]t_0,t]^c}] \leq \delta_0(t_0), \forall t > t_0 + T(t_0,\varepsilon)$, then

$$E[|X_t| \mid \mathcal{F}_{]t_0,t]^c}] < \varepsilon, \text{ a.s., for all } t \geq t_0 + T(t_0,\varepsilon);$$

(iv) conditional quasi-uniformly-asymptotically stable if, both δ and T in (iii) are independent of t_0;

(v) conditional asymptotically stable, if it is conditional stable and if there exists a $\delta_0(t_0) > 0$ such that

$$E[|x_0| \mid \mathcal{F}_{]t_0,t]^c}] \leq \delta_0(t_0), \forall t > t_0,$$

implies $E\left[|X_t| \mid \mathcal{F}_{]t_0,t]^c}\right] \to 0$ a.s. as $t \to \infty$.

(vi) conditional uniformly asymptotically stable, if it is conditional uniformly stable and quasi-uniformly asymptotically stable.

Definition 3.2 *Equations (1.1) is said to be conditional weak exponential stable , if for* *any solution X_t to (1.1) there exists a wedge function λ such that, given a constant $K > 0$, there are constants $\delta(K) > 0$ and $K' > 0$, with*

$$E[\lambda(|X_t|) \mid \mathcal{F}_{]t_0,t]^c}] \leq Ke^{-K'(t-t_0)}, \text{a.s.} \quad \forall t > t_0,$$

whenever $E[\lambda(|x_0|) \mid \mathcal{F}_{]t_0,t]^c}] \leq \delta(K), \forall t > t_0$.

Theorem 3.3 *Suppose the conditions in theorem 2.2 hold. Let X_t be any solution of (1.1) such that $F(X_t) \in \mathcal{D}^{1,4}$ and $G_i(H_t, X_s) \in \mathcal{L}^{2,p}$ for some $p > 4$. Moreover, assume that there exists a Lyapunov function $V \in C^2$ with bounded derivatives on $\{x \in R^d : |x| < M\}$ which is convex for each component x_i $(i = 1, \cdots, d)$ and satisfies the following conditions:*

(i) $V(0) = 0$;

(ii) $a(|x|) \leq V(x)$, where $a(t)$ is a continuous increasing and positive definite function on R_+;

(iii) $LV(X_t) \leq 0$, where the operator L is defined by

$$LV = (V', F) + \frac{1}{2}\sum_{j=1}^{k}(V''2D_t^j x_0 + 2V''\int_{t_0}^{t} D_t^j F \mathrm{d}s + V''G_j, G_j). \tag{1}$$

Then equations (1.1) is conditionally stable.

Proof Let $X_t = X_t(t_0, x_0)$ denote any solution of (1.1). From the conditions of the theorem, we can apply Itô formula to the functional V. Therefore

$$V(X_t) - V(x_0) = \int_{t_0}^{t}(V'(X_s), F(X_s))\mathrm{d}s + \sum_{j=1}^{k}\int_{t_0}^{t}(V'(X_s), G_j(H_t, X_s))\delta W_s^j +$$
$$\frac{1}{2}\sum_{j=1}^{k}\int_{t_0}^{t}(V''(X_s)(\nabla^j X)_s, G_j(H_t, X_s))\mathrm{d}s, \tag{2}$$

where

$$(\nabla^j X)_s = 2D_s^j x_0 + 2\int_{t_0}^{s} D_s^j F(X_u)\mathrm{d}u + 2\sum_{i=1}^{k}\int_{t_0}^{s} D_s^j G_i(H_s, X_u)_u \delta W_u^j + G_j.$$

Taking conditional expectation on both sides of (3.2), we get

$$E[V(X_t) \mid \mathcal{F}_{]t_0,t]^c}] - E[V(x_0) \mid \mathcal{F}_{]t_0,t]^c}]$$
$$= E[\int_{t_0}^{t} LV(X_s) \mid \mathcal{F}_{]t_0,t]^c}]\mathrm{d}s + \sum_{j=1}^{k} E[\int_{t_0}^{t}(V'(X_s), G_j(H_t, X_s))\delta W_s^j \mid \mathcal{F}_{]t_0,t]^c}] +$$
$$\sum_{i,j=1}^{k} E[\int_{t_0}^{t}(V''(X_s)\int_{t_0}^{s} D_s^j G_i(H_s, X_r)\delta W_r^i, G_j(H_t, X_s))\mathrm{d}s \mid \mathcal{F}_{]t_0,t]^c}].$$

It is easy to see (cf. [10]) that for each j in the second term on the right hand side of last equality, we have

$$E[\int_{t_0}^{t}(V'(X_s), G_j(H_t, X_s))\delta W_s^j \mid \mathcal{F}_{]t_0,t]^c}] = 0, j = 1, \cdots, k; \tag{3}$$

and for each i and j in the last term, we have:

$$
\begin{aligned}
&E[\int_{t_0}^{t}(V''(X_s)\int_{t_0}^{s}D_s^jG_i(H_s,X_r)\delta W_r^i, G_j(H_t,X_s))\mathrm{d}s \mid \mathcal{F}_{]t_0,t]^c}] \\
&= \int_{t_0}^{t}E[(V''(X_s)\int_{t_0}^{s}D_s^jG_i(H_s,X_r)\delta W_r^i, G_j(H_t,X_s)) \mid \mathcal{F}_{]t_0,t]^c}]\mathrm{d}s \\
&= \int_{t_0}^{t}E[(V''(X_s)\int_{t_0}^{t}D_s^jG_i(H_s,X_r)1_{[0,s]}\delta W_r^i, G_j(H_t,X_s)) \mid \mathcal{F}_{]t_0,t]^c}]\mathrm{d}s \\
&= \int_{t_0}^{t}E[(V''(X_s)\int_{t_0}^{t}D_s^jG_i(H_s,X_r)1_{[t_0,s]}, G_j(H_t,X_s))\delta W_r^i \mid \mathcal{F}_{]t_0,t]^c}]\mathrm{d}s \\
&= \int_{t_0}^{t}E[\int_{t_0}^{t}(V''(X_s)D_s^jG_i(H_s,X_r)1_{[t_0,s]}, G_j(H_t,X_s))\delta W_r^i \mid \mathcal{F}_{]t_0,t]^c}]\mathrm{d}s \\
&= 0. \qquad (4.)
\end{aligned}
$$

Therefore we have

$$E[V(X_t) \mid \mathcal{F}_{]t_0,t]^c}] - E[V(x_0) \mid \mathcal{F}_{]t_0,t]^c}] = E[\int_{t_0}^{t} LV(X_s) \mid \mathcal{F}_{]t_0,t]^c}]\mathrm{d}s. \qquad (5)$$

Given any $\varepsilon > 0$, we have

$$a(\varepsilon) \leq V(x)$$

for $x \in R^d$ such that $|x| = \varepsilon$. For a fixed $t_0 \in R_+$, since V is continuous, we can choose a $\delta(\varepsilon) > 0$ such that $|x| < \delta$ implies $V(x) < a(\varepsilon)$. Suppose equations (1.1) is not conditionally stable, then there exists a solution $x(t, t_0, x_0)$ of (1.1) such that $E[|x_0| \mid \mathcal{F}_{]t_0,t]^c}] \leq \delta(t_0, \varepsilon)$, for all $t \geq t_0$ satisfies $E[|X_{t_1}| \mid \mathcal{F}_{]t_0,t_1]^c}] \geq \varepsilon$ for some t_1. By using condition (iii), (3.5) and Jensen inequality, we have

$$a(\varepsilon) \leq E[V(X_{t_1}) \mid \mathcal{F}_{]t_0,t_1]^c}] \leq E[V(x_0) \mid \mathcal{F}_{]t_0,t_1]^c}] \leq V(E[x_0 \mid \mathcal{F}_{]t_0,t]^c}]) < a(\varepsilon) \ \text{ a.s.}$$

because $|E[x_0 \mid \mathcal{F}_{]t_0,t]^c}]| \leq E[|x_0| \mid \mathcal{F}_{]t_0,t]^c}] \leq \delta(t_0, \varepsilon)$. This is a contradiction, hence we complete the proof.

Remark From the proof of theorem 3.3, it is easy to see that the δ above does not depend on t_0. Hence equations (1.1) is conditional uniform stable under the conditions of theorem 3.3.

Theorem 3.4 *Suppose the conditions in theorem 2.2 hold. Let X_t be any solution of (1.1) such that $F(X_t) \in \mathcal{D}^{1,4}$ (i.e. each component of $F(X_t)$ in $\mathcal{D}^{1,4}$) and $G_i(H_t, X_s) \in \mathcal{L}^{2,p}$ (i.e. each component of $G_i(H_t, X_s)$ in $\mathcal{L}^{2,p}$ for some $p > 4$. Moreover, assume that there exists a Lyapunov function $V \in C^2$ with bounded derivatives on $\{x \in R^d : |x| < M\}$ which is convex for each component x_i $(i = 1, \cdots, d)$ and satisfies the following conditions:*

(i) $V(0) = 0$;

(ii) $a(|x|) \leq V(x) \leq b(|x|)$, where $a(t)$ and $b(t)$ are continuous increasing positive definite functions on R_+ and $a(t)$ is convex;

(iii) $LV(x) \leq -c(|x|)$, where $c(t)$ is a continuous function on the interval $[0, M]$ and is positive definite.

Then the equations (1.1) is conditional uniformly asymptotically stable.

Proof By theorem 3.3 and its remark we know that equations (1.1) is conditional uniformly stable, and hence, there exists a $\delta_0 > 0$, such that if $t_0 \in R_+$ and $E[|x_0| \mid \mathcal{F}_{]t_0,t]^c}] < \delta_0, \forall t > t_0$, then

$$E[|X_t| \mid \mathcal{F}_{]t_0,t]^c}] < M \quad \text{a.s.}, \quad \forall t > t_0.$$

Moreover, from the conditional stability of equations (1.1), for any $\varepsilon > 0 (\varepsilon < M)$ there exists a $\delta(\varepsilon) > 0$ such that for $t_0 \in R_+$ and $E[|x_0| \mid \mathcal{F}_{]t_0,t]^c}] < \delta(\varepsilon), \forall t > t_0$

$$E[|X_t| \mid \mathcal{F}_{]t_0,t]^c}] < \varepsilon \quad \text{a.s.}, \quad \forall t > t_0.$$

It will be shown that any solution $X_t(t_0, x_0)$ of (1.1) such that $t_0 \in R_+$, $E[|x_0| \mid \mathcal{F}_{]t_0,t]^c}] < \delta_0, \forall t > t_0$ implies $E[|X_{t_1}| \mid \mathcal{F}_{]t_0,t_1]^c}] < \delta(\varepsilon)$ a.s. at some $t_1 \geq t_0$. Suppose it is not true, then for all $t \geq t_0$

$$\delta(\varepsilon) \leq E[|X_t| \mid \mathcal{F}_{]t_0,t]^c}] < M.$$

From condition (iii) and (3.5) , there exist a $\gamma > 0$

$$E[V(X_t(t_0, x_0) \mid \mathcal{F}_{]t_0,t]^c}] \leq E[V(x_0) \mid \mathcal{F}_{]t_0,t]^c}] - \gamma(t - t_0) \quad \text{a.s.}$$

If $t > t_0 + T$, $T = [b(\delta_0) - a(\delta)]/\gamma$, since

$$E[V(x_0) \mid \mathcal{F}_{]t_0,t]^c}] \leq b(\delta_0),$$

we have

$$E[V(x_0) \mid \mathcal{F}_{]t_0,t]^c}] - \gamma(t - t_0) < a(\delta) \quad \text{a.s..}$$

Hence

$$E[V(X_t(t_0, x_0) \mid \mathcal{F}_{]t_0,t]^c}] < a(\delta)$$

which contradicts

$$E[V(X_t(t_0, x_0) \mid \mathcal{F}_{]t_0,t]^c}] \geq a(\delta) \quad \text{a.s..}$$

Thus, at some t_1 such that $t_0 \leq t_1 \leq t_0 + T$, we have

$$E[|X_{t_1}| \mid \mathcal{F}_{]t_0,t_1]^c}] < \delta(\varepsilon) \quad \text{a.s..}$$

Therefore, if $t \geq t_0 + T$, we have

$$E[|X_{t_1}| \mid \mathcal{F}_{]t_0,t_1]^c}] < \varepsilon,$$

where T only depends on ε, i.e. the equations (1.1) is conditional quasi-uniformly asymptotically stable. Thus, by definition the proof is completed.

Theorem 3.5 *Suppose the conditions in theorem 2.2 hold. Let X_t be any solution of (1.1) such that $F(X_t) \in \mathcal{D}^{1,4}$ (i.e. each component of $F(X_t)$ in $\mathcal{D}^{1,4}$) and $G_i(H_t, X_s) \in \mathcal{L}^{2,p}$ (i.e. each component of $G_i(H_t, X_s)$ in $\mathcal{L}^{2,p}$ for some $p > 4$. Moreover, assume that there exists a functional $V \in C_b^2(R^d; R_+)$ and a wedge function $\lambda \in C_0(R_+; R_+)$ (cf.[16]) such that*

$$\lambda(|x|) \leq V(x) \leq k_1 \lambda(|x|) \tag{6}$$

and

$$LV(x) \leq -k_2\lambda(|x|). \tag{7}$$

Then the equations (1.1) is conditional weak exponential stable.

Proof Let $X_t = X_t(t_0, x_0)$ denote any solution of (1.1). From the conditions of the theorem, we can apply Itô formula to the functional V. Therefore

$$V(X_t) - V(x_0) = \int_{t_0}^{t}(V'(X_s), F(X_s))\mathrm{d}s + \sum_{j=1}^{k}\int_{t_0}^{t}(V'(X_s), G_j(H_t, X_s))\delta W_s^j +$$
$$\frac{1}{2}\sum_{j=1}^{k}\int_{t_0}^{t}(V''(X_s)(\nabla^j X)_s, G_j(H_t, X_s))\mathrm{d}s.$$

Taking conditional expectation on both sides of (3.8), we get

$$\begin{aligned}
&E[V(X_t) \mid \mathcal{F}_{]t_0,t]^c}] - E[V(x_0) \mid \mathcal{F}_{]t_0,t]^c}] \\
&= E[\int_{t_0}^{t} LV(X_s) \mid \mathcal{F}_{]t_0,t]^c}]\mathrm{d}s + \sum_{j=1}^{k} E[\int_{t_0}^{t}(V'(X_s), G_j(H_t, X_s))\delta W_s^j \mid \mathcal{F}_{]t_0,t]^c}] + \\
&\sum_{i,j=1}^{k} E[\int_{t_0}^{t}(V''(X_s). \int_{t_0}^{s} D_s^j G_i(H_s, X_r)\delta W_r^i, G_j(H_t, X_s))\mathrm{d}s \mid \mathcal{F}_{]t_0,t]^c}].
\end{aligned}$$

From (3.3) and (3.4), we have

$$E[\int_{t_0}^{t}(V'(X_s), G_j(H_t, X_s))\delta W_s^j \mid \mathcal{F}_{]t_0,t]^c} = 0, j = 1, \cdots, k,$$

and for each i and j in the last term, we have:

$$E[\int_{t_0}^{t}(V''(X_s)\int_{t_0}^{s} D_s^j G_i(H_s, X_r)\delta W_r^i, G_j(H_t, X_s))\mathrm{d}s \mid \mathcal{F}_{]t_0,t]^c}] = 0.$$

Hence

$$E[V(X_t) \mid \mathcal{F}_{]t_0,t]^c}] = E[V(x_0) \mid \mathcal{F}_{]t_0,t]^c}] + E[\int_{t_0}^{t} LV(X_s)\mathrm{d}s \mid \mathcal{F}_{]t_0,t]^c}]$$

From conditions (3.6) and (3.7), we obtain

$$\begin{aligned}
E[\lambda(|X_t|) \mid \mathcal{F}_{]t_0,t]^c}] &\leq k_1 E[\lambda(|x_0|) \mid \mathcal{F}_{]t_0,t]^c}] - k_2 E[\int_{t_0}^{t} \lambda(|X_s|)\mathrm{d}s \mid \mathcal{F}_{]t_0,t]^c}] \\
&= k_1 E[\lambda(|x_0|) \mid \mathcal{F}_{]t_0,t]^c}] - k_2 \int_{t_0}^{t} E[\lambda(|X_s|) \mid \mathcal{F}_{]t_0,t]^c}]\mathrm{d}s \quad \text{a.s.}.
\end{aligned}$$

Applying Gollwitzer's inequality (cf. Corollary 1.12 in [2]), we have

$$E[\lambda(|X_t|) \mid \mathcal{F}_{]t_0,t]^c}] \leq k_1 E[\lambda(|x_0|) \mid \mathcal{F}_{]t_0,t]^c}]e^{-k_1^{-1}k_2(t-t_0)} \quad \text{a.s.}.$$

Consequently,

$$E[\lambda(|X_t|) \mid \mathcal{F}_{]t_0,t]^c}] \le Ke^{-K'(t-t_0)}, \quad \text{a.s.} \quad \forall t > t_0,$$

whenever $E\left[\lambda(|x_0|) \mid \mathcal{F}_{]t_0,t]^c}\right] \le \delta, \forall t > t_0$. That is, the equations (1.1) is conditional weak exponential stable.

Remark. Theorem 3.5 holds for the special case when the wedge function $\lambda(s) = s^p, s \in R_+, p > 0$. In this case, we call the (1.1) conditional p-stable. In the case $p = 2$, we call it conditional mean square stable.

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具有时进核的随机 Volterra 方程的条件稳定性

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摘 要: 本文引入了关于具有时进核的随机 Volterra 方程条件稳定性的若干概念，并通过李雅普诺夫函数给出此类方程条件稳定的若干判据.