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Unretractivity and End-Regularity of a Graph *

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Abstract: In this paper, a relationship among unretractivity, E-H-unretractivity and end-regularity of a graph is described.

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Introduction The graph monoid is a generalization of the graph group. The main purpose of the investigation of graph monids is to reveal interconnections between graph theory and semigroup theory. [1] may serve as a survey in this line. Regularity and various unretractivities are among the major topics in this field (cf. [2-6]). In this paper, a relationship among unretractivity, E-H-unretractivity and end-regularity of a graph is described.

1. Preliminaries

Our graphs will be finite, undirected simple ones. If G is a graph, we denote by $V(G)$ and $E(G)$ its vertex set and edge set respectively. A graph H is called a subgraph of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. Moreover, if for any $a, b \in V(H)$, $\{a,b\} \in E(H)$ if and only if $\{a,b\} \in E(G)$, then we call H an induced subgraph of G. A graph G is called an empty graph if $E(G) = \emptyset$. We denote by K_n (resp. $\overline{K_n}$) a complete graph (resp. an empty graph) with n vertices. Suppose a graph $G \neq K_1$ and $a \in G$, if $\{a,b\} \notin E(G)$ for any $b \in G$, then a is called an *isolated* vertex of G. Let G_1 and G_2 be two graphs with disjoint vertex sets. The union of G_1 and G_2, denoted by $G_1 \cup G_2$, is a graph such that $V(G_1 \cup G_2) = V(G_1) \cup V(G_2)$ and $E(G_1 \cup G_2) = E(G_1) \cup E(G_2)$.

Let G and H be graphs. A homomorphism $f : G \to H$ is a vertex-mapping $V(G) \to V(H)$ which preserves adjacency, i.e., for any $a, b \in V(G), \{a,b\} \in E(G)$ implies

$$\{f(a), f(b)\} \in E(H).$$

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A homomorphism from G to itself is called an endomorphism of G. An endomorphism f is called a strong endomorphism if $\{f(a), f(b)\} \in E(G)$ implies that $\{a, b\} \in E(G)$ for any $a, b \in V(G)$. A bijective endomorphism of a graph G is called an automorphism of G. By $\mathrm{End}(G)$, $s\mathrm{End}(G)$ and $\mathrm{Aut}(G)$ we denote the set of endomorphisms, strong endomorphisms and automorphisms of graph G respectively. It is well-known that $End(G)$ is a monoid (a monoid is a semigroup with an identity element) and $\mathrm{Aut}(G)$ is a group with respect to the composition of mappings. They are often simply called the monoid of G and the group of G respectively. Let G be a graph, $a \in V(G)$ and $f \in \mathrm{End}(G)$. Denote $f^{-1}(a) := \{b \in V(G) | f(b) = a\}$. If A is a subgraph of G, we will denote by $f|_A$ the restriction of f on $V(A)$, by $f(V(A))$ the vertex set $\{f(x) | x \in V(A)\}$ and by id_A the identity mapping from $V(A)$ to itself. An endomorphism f is said to be half-strong if $\{f(a), f(b)\} \in E(G)$ implies that there exist $c \in f^{-1}(f(a))$ and $d \in f^{-1}(f(b))$ such that $\{c, d\} \in E(G)$ (cf.[6]). The set of half-strong endomorphisms of a graph G is denoted by $h\mathrm{End}(G)$. Clearly, $\mathrm{Aut}(G) \subseteq s\mathrm{End}(G) \subseteq h\mathrm{End}(G) \subseteq \mathrm{End}(G)$. A graph G is said to be unretractive (resp. E-H-unretractive and E-S-unretractive) if $\mathrm{End}(G) = \mathrm{Aut}(G)$ (resp. $\mathrm{End}(G) = h\mathrm{End}(G)$ and $\mathrm{End}(G) = s\mathrm{End}(G)$).

Let f be an endomorphism of a graph G. A subgraph of G is called the endomorphic image of G under f, denoted by I_f, if $V(I_f) = f(V(G))$ and $\{f(a), f(b)\} \in E(I_f)$ if and only if there exist $c \in f^{-1}(f(a))$ and $d \in f^{-1}(f(b))$ such that $\{c, d\} \in E(G)$, where $a, b, c, d \in V(G)$. By ρ_f we denote the equivalence relation on $V(G)$ induced by f, i.e., for $a, b \in V(G)$, $(a, b) \in \rho_f$ if and only if $f(a) = f(b)$. Denote by $[a]_{\rho_f}$ the equivalence class of $a \in V(G)$ with respect to ρ_f. A graph, denoted by G/ρ_f, is called the *factor graph* of G under ρ_f, if $V(G/\rho_f) = V(G)/\rho_f$ and $\{[a]_{\rho_f}, [b]_{\rho_f}\} \in E(G/\rho_f)$ if and only if there exist $c \in [a]_{\rho_f}, d \in [b]_{\rho_f}$ such that $\{c, d\} \in E(G)$(cf.[4]).

An element a of a semigroup S is called regular if there exists x in S such that $axa = a$. If every element of S is regular, S is called regular. In [1] and [7], it is proved that $s\mathrm{End}(G)$ is a regular monoid. If $a^2 = a$, then a is called an idempotent of S. A graph G with $\mathrm{End}(G)$ being regular is said to be end-regular. For undefined concepts in this paper such as connected graph and semigroup, readers are refered to [8] and [9].

The following results quoted from the references will be used later.

Proposition 1.1[6.Proposition2.2] *Let G be a graph. Idempotents of* $\mathrm{End}(G)$ *are elements of* $h\mathrm{End}(G)$.

Proposition 1.2[10.Proposition1.1(ii)] *Let G be a graph and let $f \in \mathrm{End}(G)$. Then $f \in h\mathrm{End}(G)$ if and only if I_f is an induced subgraph of G.*

Proposition 1.3[4.Theorem2.5] *Let G be a graph and let $f \in \mathrm{End}(G)$. Then f is regular if and only if there exist idempotents $g, h \in \mathrm{End}(G)$ such that $\rho_g = \rho_f$ and $I_h = I_f$.*

2. The main results

In this section, we will derive the main result (Theorem 2.6), which describes a relationship among unretractivity, E-H-unretractivity and end-regularity of a graph. First, we need some lemmas.

Lemma 2.1 *Let G be a graph.*

(1) If $f \in \mathrm{End}(G)$ is regular, then $f \in h\mathrm{End}(G)$;

(2) If G is end-regular, then G is E-H-unretractive.

Proof (1) Since f is regualr, then there exists an idempotent $h \in \mathrm{End}(G)$ such that $I_h = I_f$ by Proposition 1.3. Using Proposition 1.1, we have $h \in h\mathrm{End}(G)$ and so I_h is an induced subgraph of G by Proposition 1.2. Thus I_f is an induced subgraph of G, which implies $f \in h\mathrm{End}(G)$. (2) follows directly from (1). □

Lemma 2.2 *Let G be an unretractive graph such that $G \neq K_1$. Then there does not exist any isolated vertices in G.*

Proof Suppose there exists an isolated vertex $a \in V(G)$. Since $G \neq K_1$, $V(G) \setminus \{a\} \neq \emptyset$. Let $b \in V(G) \setminus \{a\}$ and let f be a mapping from $V(G)$ to itself with $f(a) = b$ and $f(x) = x$ for any $x \in V(G) \setminus \{a\}$. It is routine to check that $f \in \mathrm{End}(G) \setminus \mathrm{Aut}(G)$, which implies G is not unretractive. □

Lemma 2.3 *Let G be an unretractive graph such that $G \neq K_1$, and let $f \in \mathrm{End}(G \bigcup \overline{K}_n)$. Then*

(1) $f|_G \in \mathrm{Aut}(G)$, and I_f is an induced subgraph of $G \bigcup \overline{K}_n$;

(2) For any $[a]_{\rho_f} \in V((G \bigcup \overline{K}_n)/\rho_f)$, $|[a]_{\rho_f} \bigcap V(G)| = 1$ or $|[a]_{\rho_f} \bigcap V(G)| = 0$;

(3) There exists an idempotent $g \in \mathrm{End}(G \bigcup \overline{K}_n)$ such that $\rho_g = \rho_f$.

Proof (1) Let $a \in V(G)$. By Lemma 2.2 a is not an isolated vertex of G and so there exists $b \in G$ such that $\{a, b\} \in E(G) = E(G \bigcup \overline{K}_n)$. Thus $\{f(a), f(b)\} \in E(G \bigcup \overline{K}_n) = E(G)$. Then $f(a) \in V(G)$ and so $f|_G \in \mathrm{End}(G)$. Noticing G is unretractive, we have $f|_G \in \mathrm{Aut}(G)$, and so it is easy to see that I_f is an induced subgraph of $G \bigcup \overline{K}_n$.

(2) Suppose there exists $[a]_{\rho_f} \in V((G \bigcup \overline{K}_n)/\rho_f)$ such that $|[a]_{\rho_f} \bigcap V(G)| \geq 2$. Let $x, y \in [a]_{\rho_f} \bigcap V(G)$ $(x \neq y)$. Then we have $f(x) = f(y)$ and $x, y \in V(G)$, which means $f|_G(x) = f|_G(y)$, a contradiction to $f|_G \in Aut(G)$, the result of (1).

(3) By (2) we may construct a mapping g from $V(G \bigcup \overline{K}_n)$ to itself by the following rule: for any $[a]_{\rho_f} \in V((G \bigcup \overline{K}_n)/\rho_f)$, if $|[a]_{\rho_f} \bigcap V(G)| = 1$, then let $[a]_{\rho_f} \bigcap V(G) = \{c\}$ and assign $g(x) = c$ for any $x \in [a]_{\rho_f}$; if $|[a]_{\rho_f} \bigcap V(G)| = 0$, then select (arbitrarily but fixedly) one vertex $c \in [a]_{\rho_f}$ and assign $g(x) = c$ for any $x \in [a]_{\rho_f}$.

Obviously, g is well-defined as a mapping. It is easy to check that $g|_G = \mathrm{id}_G$, and so $g \in \mathrm{End}(G \bigcup \overline{K}_n)$ noticing $E(G \bigcup \overline{K}_n) = E(G)$. Let $a \in V(G \bigcup \overline{K}_n)$. If $[a]_{\rho_f} \bigcap V(G) = \{c\}$ for some $c \in V(G \bigcup \overline{K}_n)$, then $c \in V(G)$ and $g(a) = c = \mathrm{id}_G(c) = g|_G(c) = g(c) = g(g(a)) = g^2(a)$. If $[a]_{\rho_f} \bigcap V(G) = \emptyset$, then there exists $c \in [a]_{\rho_f}$ such that $g(x) = c$ for any $x \in [a]_{\rho_f}$, and so $g(a) = c = g(c) = g(g(a)) = g^2(a)$. Thus g is an idempotent of $\mathrm{End}(G \bigcup \overline{K}_n)$. By the definition of g, for any $x, y \in V(G \bigcup \overline{K}_n)$, $g(x) = g(y)$ if and only if $[x]_{\rho_f} = [y]_{\rho_f}$, i.e., $g(x) = g(y)$ if and only if $f(x) = f(y)$, which implies $\rho_g = \rho_f$.

Lemma 2.4 *Let G be a graph, let K_n $(n \geq 1)$ be a complete graph with n vertices and let $\overline{K}_n$ $(n \geq 2)$ be an empty graph with n vertices. Then*

(1)[1,Theorem4.1] *$s\mathrm{End}(G)$ is regular;*

(2)[2,Table4.1.1] *K_n is unretractive;*

(3)[2,Example1.2] $\overline{K}_n$ *is E-S-unretractive without being unretractive;*

(4) $\mathrm{End}(\overline{K}_n)$ *is regular.*

Proof of (4) This follows immediately from (1) and (3).

Proposition 2.5 *Let G be an unretractive graph. Then for any $n \geq 1$, $G \bigcup \overline{K}_n$ is end-regular.*

Proof Suppose G is unretractive. If $G = K_1$, $G \bigcup \overline{K}_n = \overline{K}_{n+1}$, and so $G \bigcup \overline{K}_n$ is end-regular by Lemma 2.4(4). Now let $G \neq K_1$ and let $f \in \mathrm{End}(G \bigcup \overline{K}_n)$. Denote $A = f(V(\overline{K}_n)) \bigcap V(\overline{K}_n)$. We consider two cases: (i) $A \neq \emptyset$. Clearly $A \subseteq V(\overline{K}_n)$. Let $a \in A$ and let h be a mapping from $V(G \bigcup \overline{K}_n)$ to itself such that $h|_G = id_G$, $h|_A = \mathrm{id}_A$ and $h(x) = a$ for any $x \in V(\overline{K}_n) \setminus A$ if $V(\overline{K}_n) \setminus A \neq \emptyset$. It is easy to check that h is well-defined and $h \in \mathrm{End}(G \bigcup \overline{K}_n)$. Since for any $x \in V(\overline{K}_n) \setminus A$ (if $V(\overline{K}_n) \setminus A \neq \emptyset$), $h(x) = a = \mathrm{id}_A(a) = h|_A(a) = h(a) = h^2(x)$, h is an idempotent of $\mathrm{End}(G \bigcup \overline{K}_n)$. Then by Propositions 1.1 and 1.2, I_h is an induced subgraph of $G \bigcup \overline{K}_n$. Furthermore, by Lemma 2.3(1) we see $V(I_f) = V(G) \bigcup A = V(I_h)$, and I_f is an induced subgraph of $G \bigcup \overline{K}_n$. Therefore, $I_f = I_h$. (ii) $A = \emptyset$. Let $a \in V(G)$ and let h be a mapping from $V(G \bigcup \overline{K}_n)$ to itself such that $h|_G = \mathrm{id}_G$ and $h(x) = a$ for any $x \in V(\overline{K}_n)$. Similarly, we can check that h is an idempotent of $\mathrm{End}(G \bigcup \overline{K}_n)$ such that $I_h = I_f$. Hence, in any case there exists an idempotent $h \in \mathrm{End}(G \bigcup \overline{K}_n)$ such that $I_h = I_f$. Thus, using Lemma 2.3(3) and Proposition 1.3, we can conclude f is regular. □

Theorem 2.6 *Let G be a connected graph. Then the following three statements are equivalent:*

(1) *G is unretractive;*

(2) *For any $n \geq 1$, $G \bigcup \overline{K}_n$ is end-regular;*

(3) *For any $n \geq 1$, $G \bigcup \overline{K}_n$ is E-H-unretractive.*

Proof (1) $\Rightarrow$ (2) This is a straightforward consequence of Proposition 2.5.

(2) $\Rightarrow$ (3) This follows directly from Lemma 2.1(2).

(3) $\Rightarrow$ (1) Suppose G is not unretractive, i.e.,

$$\mathrm{End}(G) \setminus \mathrm{Aut}(G) \neq \emptyset. \text{ Let } \psi \in \mathrm{End}(G) \setminus \mathrm{Aut}(G).$$

Then $V(G) \setminus V(I_\psi) \neq \emptyset$. Since G is connected, there exist $a \in V(G) \setminus V(I_\psi)$ and $b \in V(I_\psi)$ such that $\{a, b\} \in E(G)$. Let f be a mapping from $V(G \bigcup \overline{K}_n)$ to itself such that $f(x) = a$ if $x \in V(\overline{K}_n)$ and $f(x) = \psi(x)$ if $x \in V(G)$. Noticing $E(G \bigcup \overline{K}_n) = E(G)$ and $\psi \in \mathrm{End}(G)$, we see $f \in \mathrm{End}(G \bigcup \overline{K}_n)$. By the definition of f, $a \in f(V(\overline{K}_n)) \subseteq V(I_f)$ and $b \in V(I_\psi) = \psi(V(G)) \subseteq V(I_f)$. Because $x \in V(\overline{K}_n)$ for any $x \in f^{-1}(a)$, and $y \in V(G)$ for any $y \in f^{-1}(b)$, $\{a, b\} \notin E(I_f)$. However, $\{a, b\} \in E(G \bigcup \overline{K}_n)$, and so I_f is not an induced subgraph of $G \bigcup \overline{K}_n$. Then by Proposition 1.2 $f \notin h\mathrm{End}(G \bigcup \overline{K}_n)$. Thus $G \bigcup \overline{K}_n$ is not E-H-unretractive. □

Remark 2.7 In general, if a graph G is not connected, the three statements in Theorem 2.6 are not equivalent. For examples, let $G = K_3 \bigcup K_1$. It follows from Lemma 2.4(2) and Theorem 2.6 that $\mathrm{End}(G \bigcup K_1) = \mathrm{End}(K_3 \bigcup \overline{K}_2)$ is regular, however G is not unretractive by Lemma 2.2; Let $G = K_2 \bigcup K_2$. It can be checked that $G \bigcup K_1$ is E-H-unretractive, but G is not unretractive.

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图的不可收缩性和 end- 正则性

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摘 要: 图的半群理论是图的群理论的延伸. 图的不可收缩性和 end- 正则性是其中受到普遍关注的课题. 本文揭示了两者之间的内在联系.